Forschungszentrum Telekommunikation Wien

Theory and Design of

Turbo and Related Codes

Lecture 12

Gottfried Lechner & Jossy Sayir

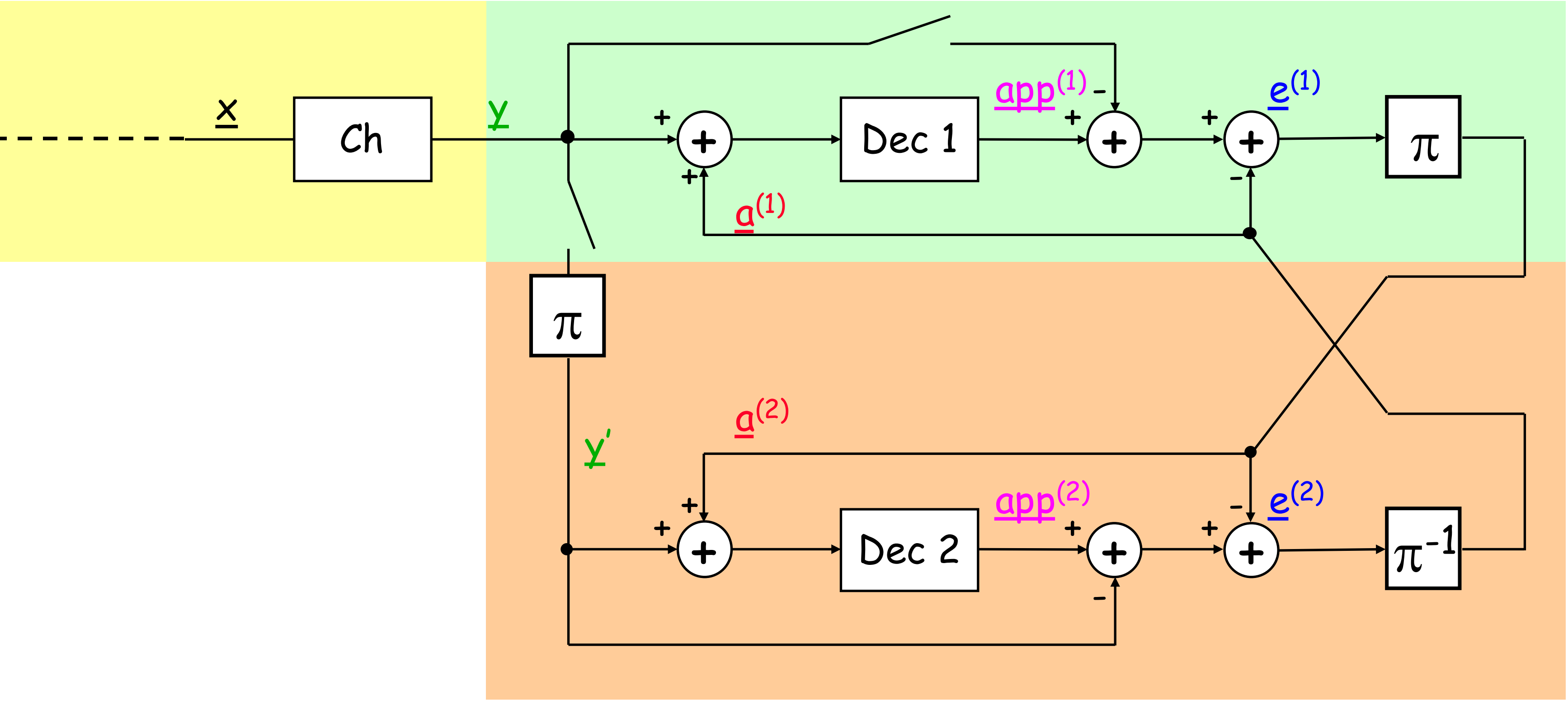
http://userver.ftw.at/~jossy/turbo/index.html

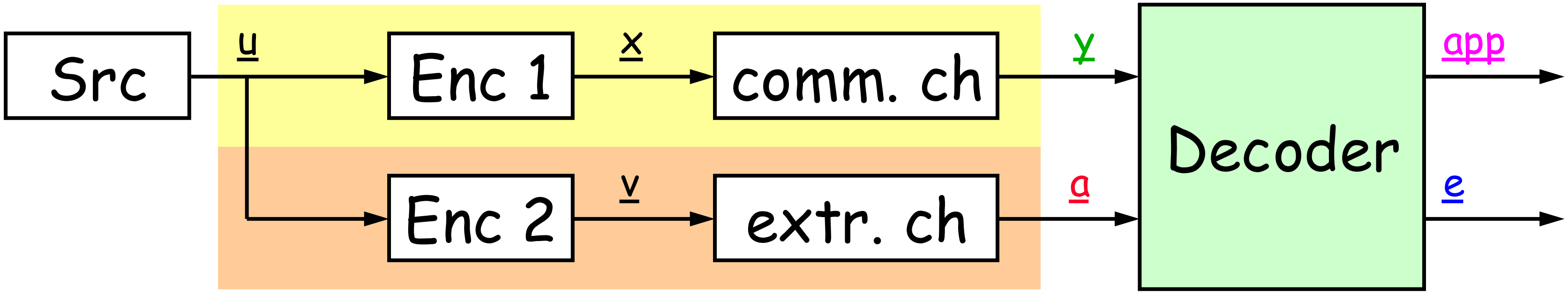
Kompetenzzentren-Programm

Last Lecture

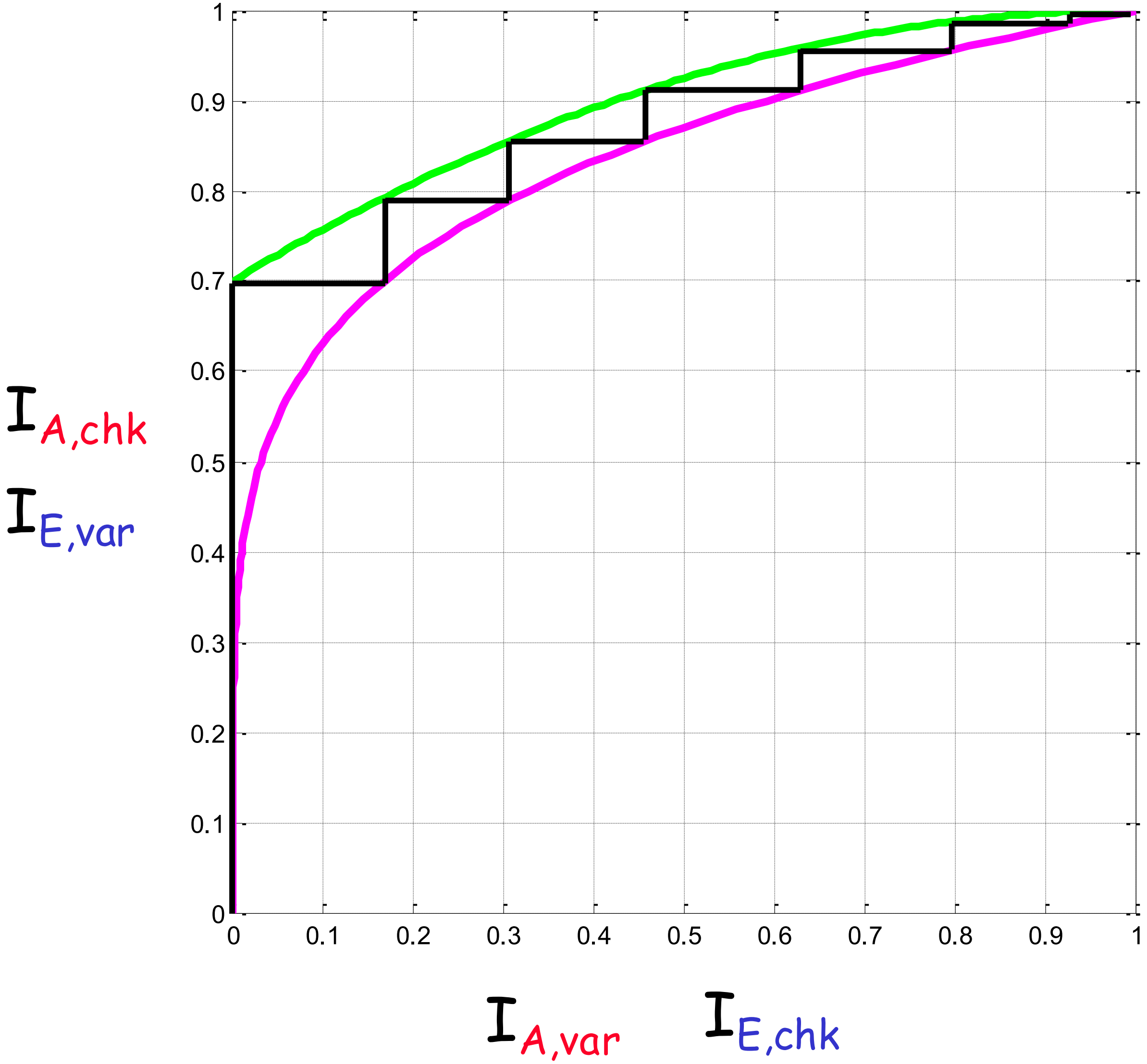
- We computed information transfer functions of the component decoders.
- We combined these functions to EXIT charts.
- We used these charts to predict the convergence behavior of the iterative decoder.
- We made some assumptions and/or simplifications.

Extrinsic Channel Model (from last lecture)





EXIT Chart of LDPC Code (from last lecture)



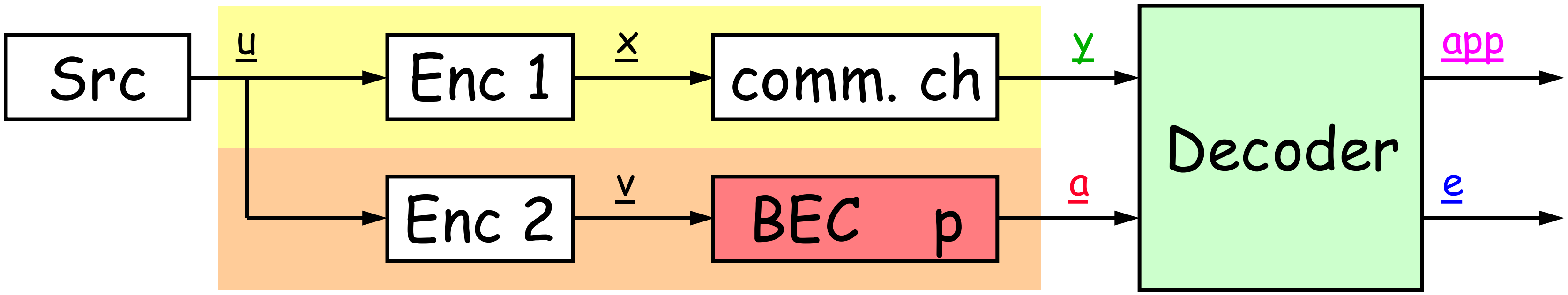
Summary of Assumptions (from last lecture)

- Messages received from the extrinsic channel are independent observations, which is only fulfilled if $N \to \infty$

- We use statistical quantities, which are only correct if $N \to \infty$

- We model extrinsic messages with an extrinsic channel. This can only be done exact for the BEC. The Gaussian assumption is an approximation.

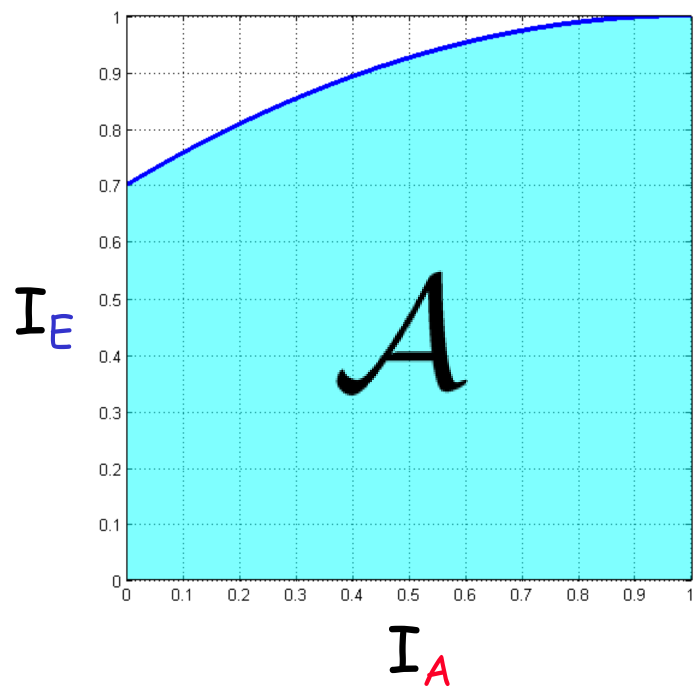
Area Property



$$m = |\underline{v}|$$

$$I_A = \frac{1}{m}\sum_{i=1}^{m} I(V_i; A_i) = 1 - p$$

$$\mathcal{A} = \int_0^1 I_E(I_A)\,dI_A = \int_0^1 I_E(p)\,dp$$



Derivation of Area Property 1

$$I_E = \frac{1}{m}\sum_{i=1}^{m} \underbrace{I(V_i; \underline{Y}\,\underline{A}_{\setminus i})}$$

$$\underbrace{H(V_i)}_{1} - \underbrace{H(V_i|\underline{Y}\,\underline{A}_{\setminus i})}$$

$$\sum_{\underline{a}_{\setminus i}} P(\underline{a}_{\setminus i}) \cdot \underbrace{H(V_i|\underline{Y}, \underline{A}_{\setminus i} = \underline{a}_{\setminus i})} \qquad \mathcal{S} = \{b = 1 \ldots m | b \neq i, A_b \neq \Delta\}$$

$$\underbrace{H(V_i|\underline{Y}, \underline{V}_{\mathcal{S}} = \underline{v}_{\mathcal{S}})} \qquad 0 \leq j = |\mathcal{S}| \leq m-1$$

$$\sum_{j=0}^{m-1} (1-p)^j \cdot p^{m-1-j} \sum_{|\mathcal{S}|=j, i \notin \mathcal{S}} H(V_i|\underline{Y}\,\underline{V}_{\mathcal{S}})$$

$$I_E = \frac{1}{m}\sum_{i=1}^{m}\left[1 - \sum_{j=0}^{m-1} (1-p)^j \cdot p^{m-1-j} \sum_{|\mathcal{S}|=j} H(V_i|\underline{Y}\,\underline{V}_{\mathcal{S}})\right]$$

$$I_E = 1 - \frac{1}{m}\sum_{i=1}^{m}\sum_{j=0}^{m-1} (1-p)^j \cdot p^{m-1-j} \sum_{|\mathcal{S}|=j} H(V_i|\underline{Y}\,\underline{V}_{\mathcal{S}})$$

Derivation of Area Property 2

$$I_E = 1 - \frac{1}{m} \sum_{i=1}^{m} \sum_{j=0}^{m-1} (1-p)^j \cdot p^{m-1-j} \sum_{|\mathcal{S}|=j} H(V_i | \underline{Y} \underline{V}_{\mathcal{S}})$$

$$\mathcal{A} = \int_0^1 1 - \frac{1}{m} \sum_{i=1}^{m} \sum_{j=0}^{m-1} (1-p)^j \cdot p^{m-1-j} \sum_{|\mathcal{S}|=j} H(V_i | \underline{Y} \underline{V}_{\mathcal{S}}) dp$$

$$\mathcal{A} = 1 - \frac{1}{m} \sum_{i=1}^{m} \sum_{j=0}^{m-1} \int_0^1 (1-p)^j \cdot p^{m-1-j} dp \sum_{|\mathcal{S}|=j} H(V_i | \underline{Y} \underline{V}_{\mathcal{S}})$$

$$\mathcal{A} = 1 - \frac{1}{m} \underbrace{\sum_{i=1}^{m} \sum_{j=0}^{m-1} \frac{1}{m\binom{m-1}{j}} \sum_{|\mathcal{S}|=j} H(V_i | \underline{Y} \underline{V}_{\mathcal{S}})}_{H(\underline{V} | \underline{Y})}$$

$$\boxed{\mathcal{A} = 1 - \frac{1}{m} H(\underline{V} | \underline{Y}) = 1 - \frac{1}{m} H(\underline{X} | \underline{Y})}$$

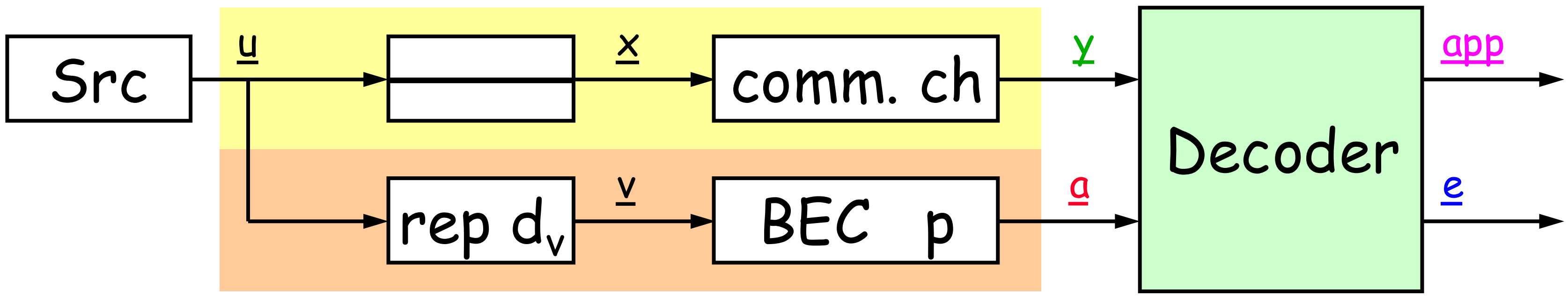
Derivation of H(V|Y)

$$\begin{aligned} H(\underline{V}) &= H(V_1) + H(V_2|V_1) + H(V_3|V_1V_2) \\ &= H(V_1) + H(V_3|V_1) + H(V_2|V_1V_3) \\ &= H(V_2) + H(V_1|V_2) + H(V_3|V_1V_2) \\ &= H(V_2) + H(V_3|V_2) + H(V_1|V_2V_3) \\ &= H(V_3) + H(V_1|V_3) + H(V_2|V_1V_3) \\ &= H(V_3) + H(V_2|V_3) + H(V_1|V_2V_3) \end{aligned}$$

$$\begin{aligned} 3! \cdot H(\underline{V}) \;&=\; 2 \cdot (H(V_1) + H(V_2) + H(V_3)) + \\ &+\; H(V_1|V_2) + H(V_1|V_3) + H(V_2|V_1) + H(V_2|V_3) + H(V_3|V_1) + H(V_3|V_2) \\ &+\; 2 \cdot (H(V_1|V_2V_3) + H(V_2|V_1V_3) + H(V_3|V_1V_2)) \end{aligned}$$

$$\sum_{i=1}^{m} \sum_{j=0}^{m-1} \frac{1}{m\binom{m-1}{j}} \sum_{|\mathcal{S}|=j} H(V_i|\underline{Y}\underline{V}_{\mathcal{S}})$$

Variable Nodes



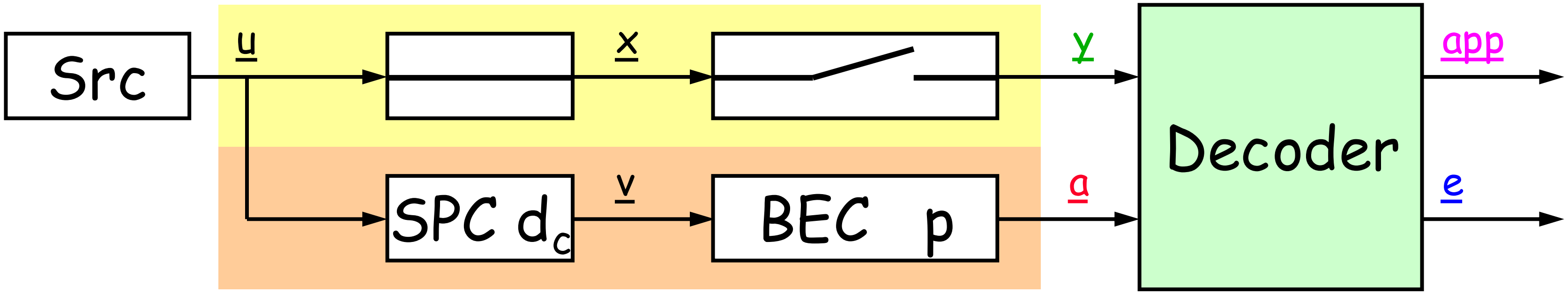
$$|\underline{u}| = k \qquad m = |\underline{v}| = k \cdot d_v$$

$$|\underline{x}| = k$$

$$\mathcal{A}_v = 1 - \frac{1}{m} H(\underline{X}|\underline{Y}) = 1 - \frac{H(\underline{X}) - I(\underline{X};\underline{Y})}{k \cdot d_v}$$

$$= 1 - \frac{k - k \cdot I(X_1;Y_1)}{k \cdot d_v} = 1 - \frac{1 - C}{d_v}$$

Check Nodes



$$|\underline{u}| = k \qquad m = |\underline{v}| = k \cdot \frac{d_c}{d_c - 1}$$
$$|\underline{x}| = k$$

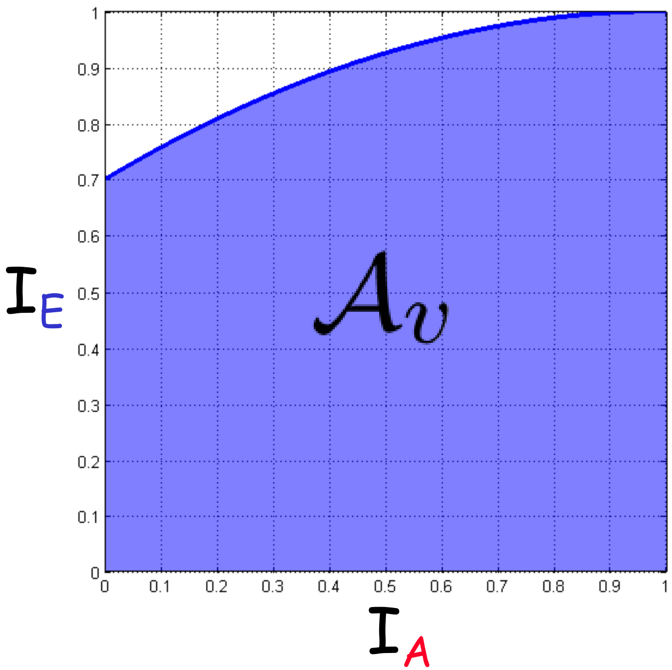
$$\mathcal{A}_c = 1 - \frac{1}{m} H(\underline{V} | \underline{Y}) = 1 - \frac{H(\underline{V}) - I(\underline{V}; \underline{Y})}{m}$$

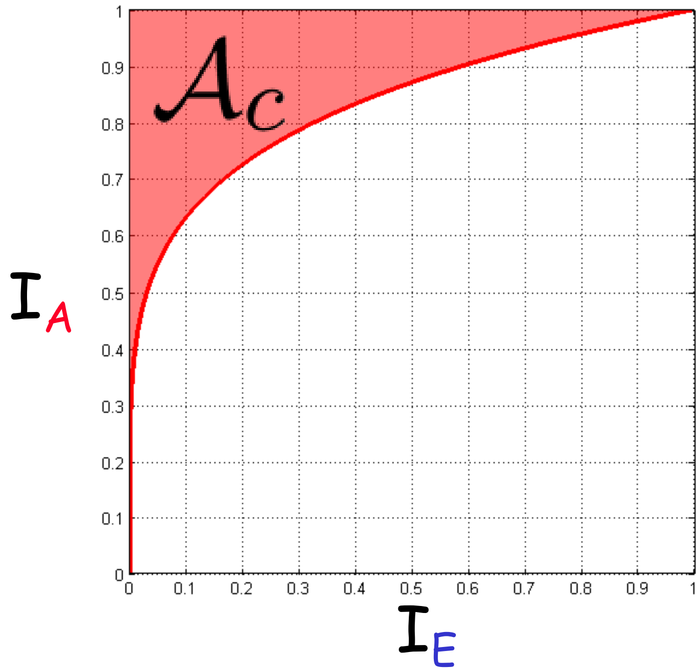
$$= 1 - \frac{k \cdot (d_c - 1)}{k \cdot d_c} = \frac{1}{d_c}$$

Area of LDPC Component Codes

$$\mathcal{A}_v = 1 - \frac{1-C}{d_v}$$

$$\mathcal{A}_c = \frac{1}{d_c}$$





Necessary condition for successful decoding:

$$1 - \mathcal{A}_v < \mathcal{A}_c$$

Consequences of Area Property

$$1 - \mathcal{A}_v < \mathcal{A}_c$$

$$1 - 1 + \frac{1 - C}{d_v} < \frac{1}{d_c}$$

$$1 - C < \frac{d_v}{d_c}$$

$$C > 1 - \frac{d_v}{d_c} = R$$

"Surprising" result:

The area property tells us that the decoder can only converge if the rate is smaller than capacity!

More Consequences...

Suppose the condition for convergence is fulfilled

$$1 - \mathcal{A}_v = \gamma \cdot \mathcal{A}_c < \mathcal{A}_c \qquad 0 \cdot \gamma < 1$$

$$\mathcal{A}_c = \frac{1}{d_c}$$

$$1 - \mathcal{A}_v = \gamma \cdot \mathcal{A}_c = \frac{1 - C}{d_v}$$

$$R = 1 - \frac{d_v}{d_c} = 1 - \frac{1 - C}{\gamma} = \frac{C - (1 - \gamma)}{\gamma} < C$$

What is the result of this inequality?

Area and Rate Loss

$$R = 1 - \frac{d_v}{d_c} = 1 - \frac{1-C}{\gamma} = \frac{C-(1-\gamma)}{\gamma} < C$$

If $\gamma \to 1$ we can transmit at rates that approach capacity.
If $\gamma < 1$ we are bounded from capacity.

$\gamma \to 1$ means that $1 - A_v = A_c$

Furthermore, the curves must not intersect.

$\Rightarrow$ The curves have to be matched.

Code design reduces to curve fitting!

Curve Fitting – Code Mixture

We only considered regular codes, where every symbol has the same properties. Therefore, averaging over all symbols is equivalent to the mutual information of an arbitrarily symbol.

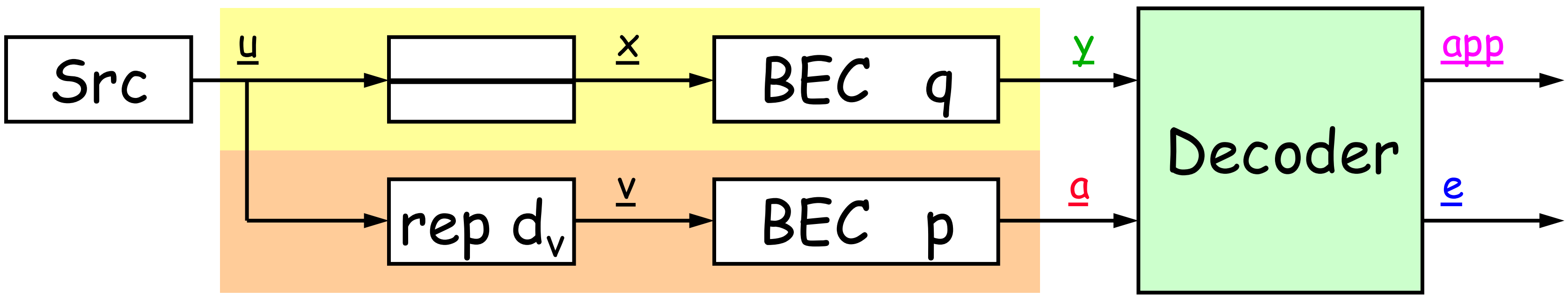
$$I_E = \frac{1}{m} \sum_{i=1}^{m} I(V_i; E_i) = I(V_1; E_1)$$

If we partition m into n_u groups $j=1 \ldots n_u$ each with length l_j, we can write I_E as

$$I_E = \sum_{j=1}^{n_u} \frac{l_j}{m} \left[\frac{1}{l_j} \sum_{i=1}^{l_j} I(V_{ji}; E_{ji}) \right] = \sum_{j=1}^{n_u} \gamma_j I_{E_j} \qquad \gamma_j = \frac{l_j}{m} = \frac{l_j}{\sum_{j=1}^{n_u} l_j}$$

The resulting EXIT function is the weighted average of the EXIT functions of the groups.

Example – Variable Mixture



70% of the variable nodes have d_v=2
30% of the variable nodes have d_v=5

$$\gamma_1 = \frac{0.7 \cdot k \cdot 2}{0.7 \cdot k \cdot 2 + 0.3 \cdot k \cdot 5} = 0.48 \qquad \gamma_2 = \frac{0.3 \cdot k \cdot 5}{0.7 \cdot k \cdot 2 + 0.3 \cdot k \cdot 5} = 0.52$$

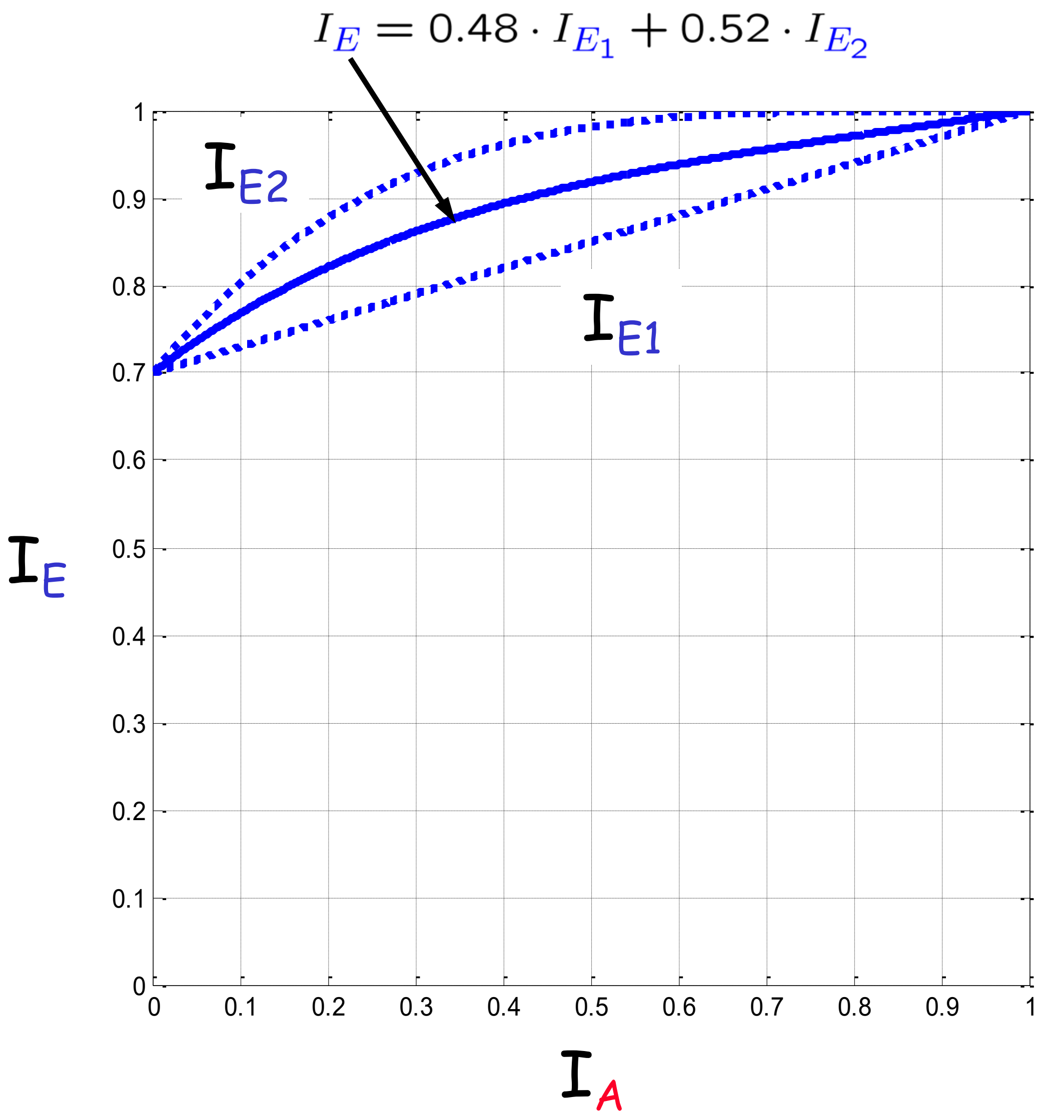
$$I_{E_j} = 1 - qp^{d_{vj}-1} \qquad I_E = \gamma_1 \cdot \left[1 - qp^{d_{v1}-1}\right] + \gamma_2 \cdot \left[1 - qp^{d_{v2}-1}\right]$$

$$I_E(p) = 1 - q \cdot \sum_{j=1}^{n_u} \gamma_j \cdot p^{d_{vj}-1}$$

This is a polynomial in p

Note that $\sum \gamma_j = 1$

Example – Variable Mixture



Curve Fitting

Lets fix the EXIT function of the check node decoder.

$$I_{Ec} = (I_{Ac})^{d_c - 1}$$

For curve fitting, we can exchange the following quantities

$$I_{Ec} = I_{Av} \qquad I_{Ev} = I_{Ac}$$

Therefore, we can write the EXIT function of the variable node decoder as the inverse EXIT function of the check node decoder.

$$I_{Av} = (I_{Ev})^{d_c - 1}$$

$$I_{Ev} = (I_{Av})^{\frac{1}{d_c - 1}} = (1 - p)^{\frac{1}{d_c - 1}}$$

Taylor Series Expansion

$$I_{Ev} = (I_{Av})^{\frac{1}{dc-1}} = (1-p)^{\frac{1}{dc-1}}$$

Assuming for example d_c=5 we can expand I_{Ev} as a Taylor series

$$I_{Ev} = 1 - \left[\frac{1}{4}p + \frac{3}{32}p^2 + \frac{7}{128}p^3 + \cdots\right]$$

Truncating the Taylor series and normalizing the coefficients to 1 results in

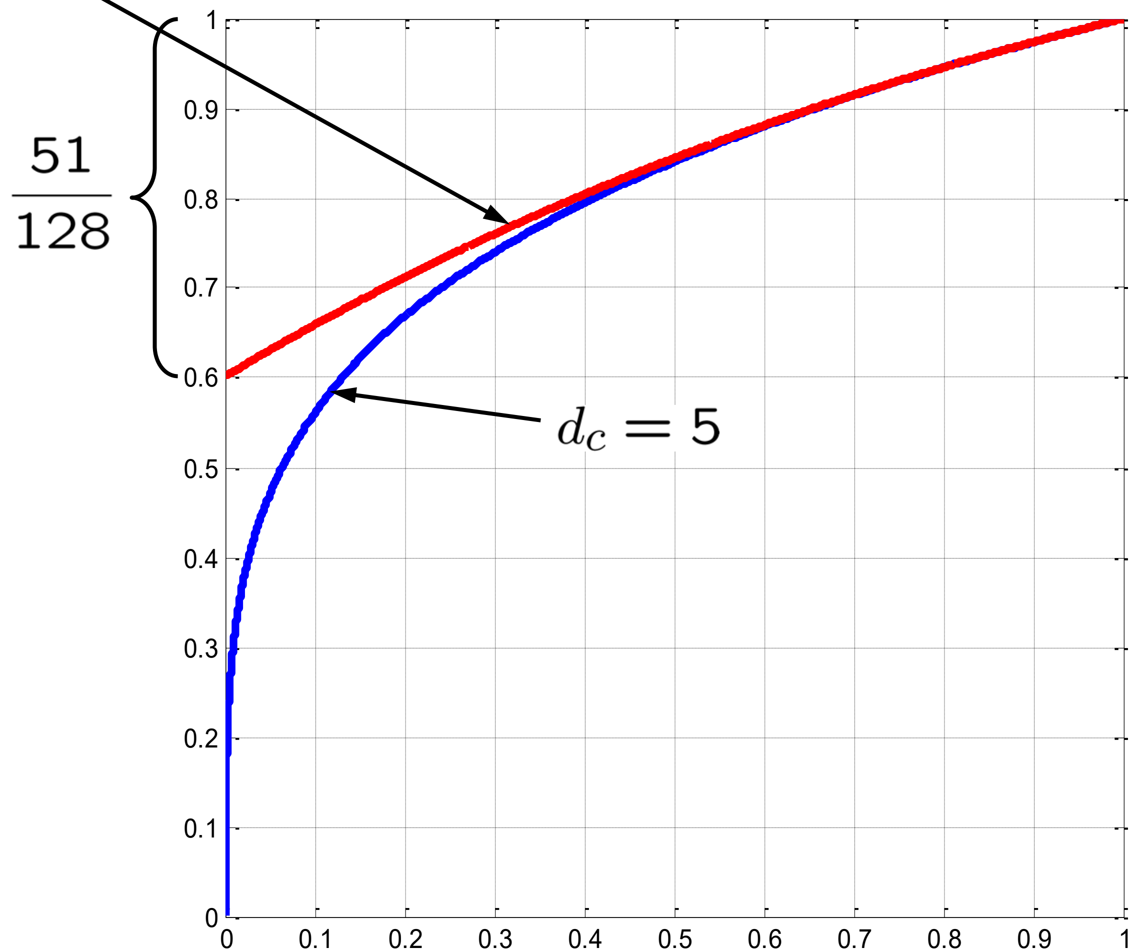
$$I_{Ev} = 1 - \frac{51}{128}\left[\frac{32}{51}p + \frac{12}{51}p^2 + \frac{7}{51}p^3\right]$$

Compare this with the transfer function of the mixture of variable nodes...

$$I_E(p) = 1 - q \cdot \sum_{j=1}^{n_u} \gamma_j \cdot p^{d_{vj}-1}$$

Curve Fitting

$$I_{Ev} = 1 - \frac{51}{128}\left[\frac{32}{51}(1 - I_{Av}) + \frac{12}{51}(1 - I_{Av})^2 + \frac{7}{51}(1 - I_{Av})^3\right]$$



Even more Consequences...

Using the same model as for the variable and check node decoder, it can be shown that the areas for a serial concatenated code with an outer code $R_{out}=k_{out}/n_{out}$ and an inner code $R_{in}=k_{in}/n_{in}$ are given by

$$\mathcal{A}_{out} = 1 - R_{out} \qquad\qquad \mathcal{A}_{in} = \frac{I(\underline{X};\underline{Y})}{n_{in} \cdot R_{in}}$$

The same necessary condition $1-\mathcal{A}_{out} < \mathcal{A}_{in}$ leads to

$$R_{out} \cdot R_{in} < \frac{I(\underline{X};\underline{Y})}{n_{in}} \leq C$$

If the inner code has rate < 1, i.e. $I(\underline{X};\underline{Y})/n_{in} < C$ then we can not achieve capacity with serial concatenated codes!

Summary

- The area under the information transfer function has some important consequences for the rate of the code and for code design.
- Capacity can only be achieved, if the curves of the component decoders are matched.
- Furthermore, serial concatenated codes can only achieve capacity if the rate of the inner encoder is ≥ 1.