[Telecommunications Research Center Vienna]

Theory and Design of

Turbo and Related Codes

Lecture 3

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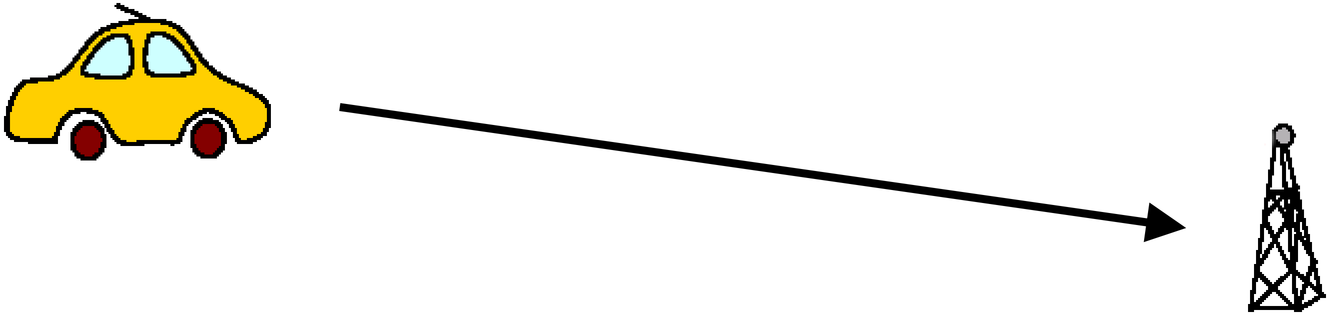
Kplus
Kompetenzzentren-Programm

The Channel



- The channel is the link between the transmitter and the receiver.
- Signals sent by the transmitter will usually be corrupted by the channel.
- The corruption of the signal has some random source.
- We use channel models that approximate real channels.

The Channel

The channel maps a symbol of the input alphabet to a symbol of the output alphabet (randomly).

This mapping can be described by the probability of a symbol at the output, given that a symbol at the input has been transmitted.

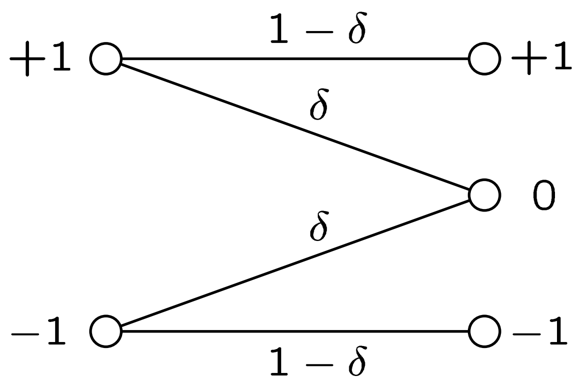
$$P(Y = y|X = x)$$

Channels with Discrete Output

Binary Erasure Channel
BEC

$$X \in \mathcal{A}_{in} = \{+1, -1\}$$

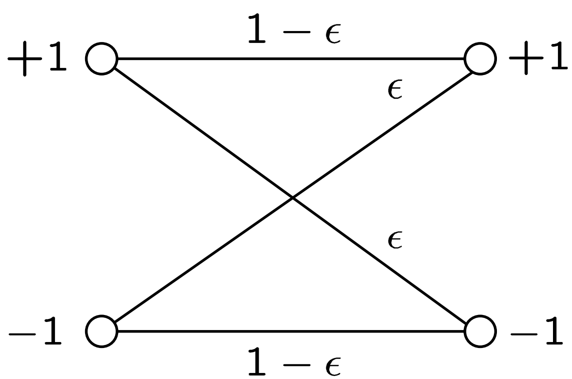
$$Y \in \mathcal{A}_{out} = \{+1, 0, -1\}$$



Binary Symmetric Channel
BSC

$$X \in \mathcal{A}_{in} = \{+1, -1\}$$

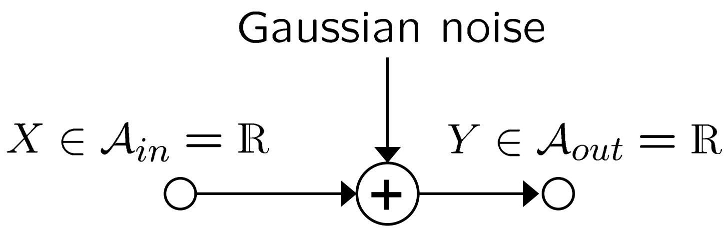
$$Y \in \mathcal{A}_{out} = \{+1, -1\}$$

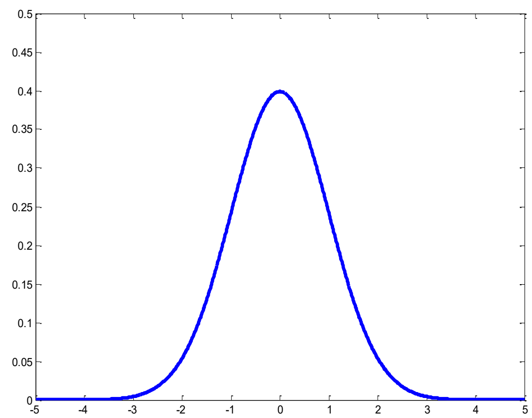


Does the decoding algorithm of lecture 1 also work for a BSC?

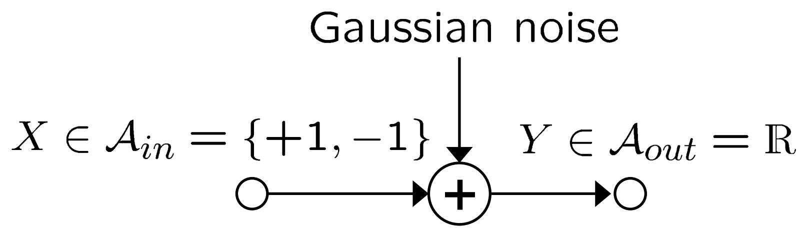
Channels with Continuous Output

Additive White Gaussian Noise AWGN





Binary Input AWGN BIAWGN

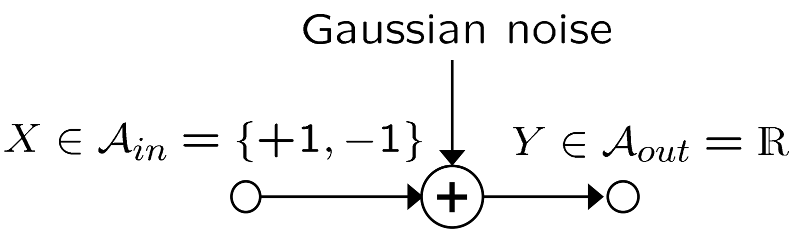


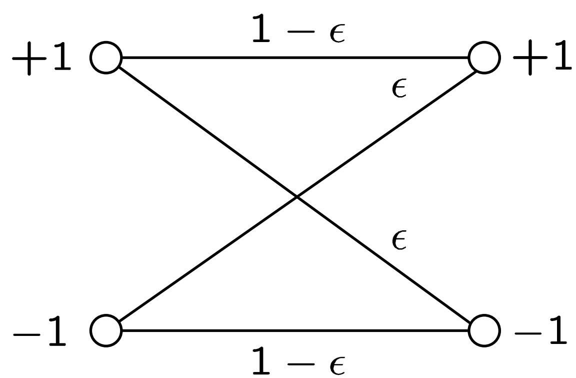
$$P(Y = y | X = x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(y-x)^2}{2\sigma^2}}$$

Carl Friedrich Gauss 1777-1855

BIAWGN vs. BSC

A BIAWGN channel with “hard decision” becomes a BSC.

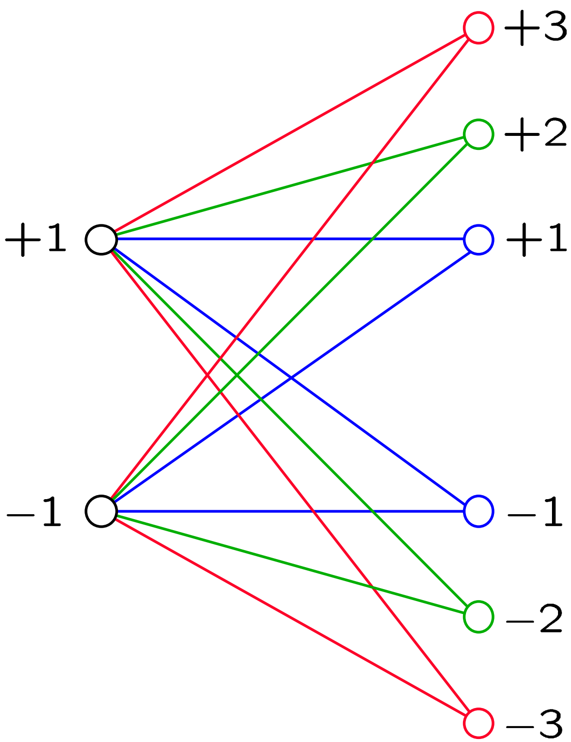


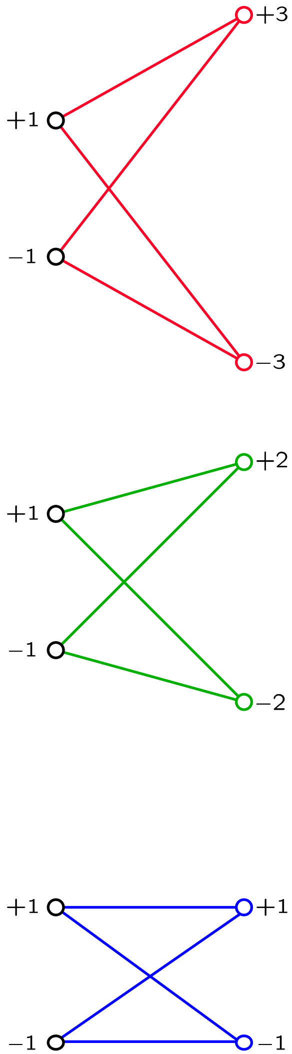


$$P(Y=y|X=x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(y-x)^2}{2\sigma^2}}$$

$$\epsilon = P(Y>0|X=-1) = \int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(y+1)^2}{2\sigma^2}} dy$$

Higher output Alphabet





Channels - Summary

- We use only binary input channels in this course.
- Channels are memoryless.
- Fully described by their transition probabilities.
- All channels satisfy a symmetry property

$$P(Y = y|X = x) = P(Y = -y|X = -x)$$

Abstraction of Channels

In the end we are only interested which binary digit was transmitted.

Therefore, we have to compare probabilities of +1 or -1 conditioned on our observation.

$$P(X=+1|Y=y) \gtrless P(X=-1|Y=y)$$

$$P(X=+1|Y=y) - P(X=-1|Y=y) \gtrless 0$$

$$\frac{P(X=+1|Y=y)}{P(X=-1|Y=y)} \gtrless 1$$

$$\log \frac{P(X=+1|Y=y)}{P(X=-1|Y=y)} \gtrless 0$$

Log-Likelihood Ratio

$$\mathsf{L}(X|Y=y) = \log \frac{P(X=+1|Y=y)}{P(X=-1|Y=y)}$$

$$= \log \frac{P(Y=y|X=+1)\cdot P(X=+1)}{P(Y=y|X=-1)\cdot P(X=-1)}$$

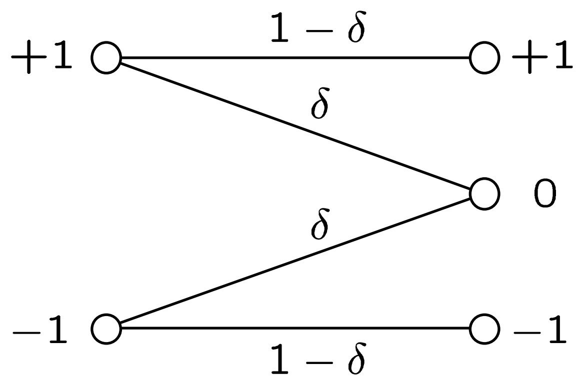
$$= \underbrace{\log \frac{P(Y=y|X=+1)}{P(Y=y|X=-1)}}_{\text{channel L-value}} + \underbrace{\log \frac{P(X=+1)}{P(X=-1)}}_{\text{a-priori L-value}}$$

$$\mathsf{L}_{X,intr}(y) = \mathsf{L}_{X,ch}(y) + \mathsf{L}_{X,apri}$$

Note: $\mathsf{L}_{X,intr}(y)$ is a only a function of y – not of x! We will omit the subscript X if it is clear from the context.

LLR of some Channels

BEC

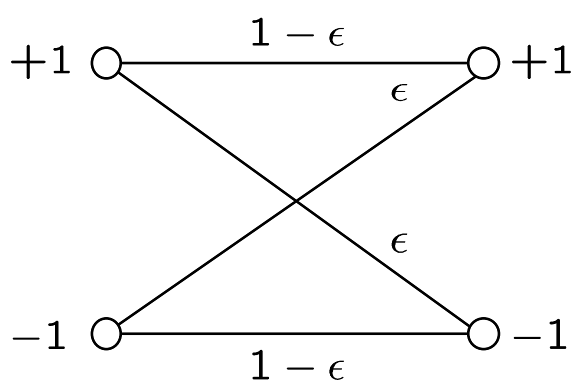


$$\mathsf{L}_{intr}(+1) = L_{apri} + \infty$$

$$\mathsf{L}_{intr}(0) = L_{apri} + 0$$

$$\mathsf{L}_{intr}(-1) = L_{apri} - \infty$$

BSC



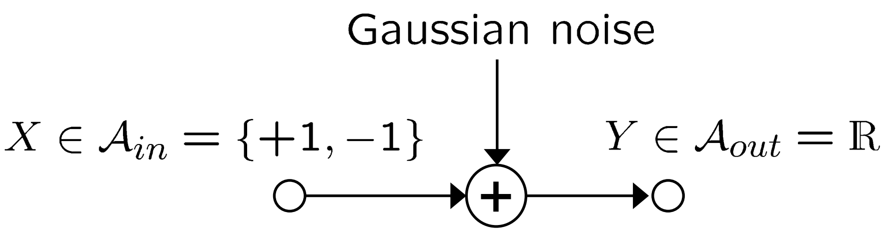
$$\mathsf{L}_{intr}(+1) = L_{apri} + \log \frac{1-\epsilon}{\epsilon}$$

$$\mathsf{L}_{intr}(-1) = L_{apri} + \log \frac{\epsilon}{1-\epsilon}$$

Note the symmetry of the channels.

LLR of some Channels

BIAWGN



$$\mathsf{L}_{ch}(y) = \log \frac{P(Y = y|X = +1)}{P(Y = y|X = -1)}$$

$$= \log \frac{\frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(y-1)^2}{2\sigma^2}}}{\frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(y+1)^2}{2\sigma^2}}}$$

$$= \frac{-(y-1)^2 + (y+1)^2}{2\sigma^2} = \frac{2y}{\sigma^2}$$

The L-Value of the channel is a scaled version of the received value y.

$$\mathsf{L}_{intr}(y) = \frac{2y}{\sigma^2} + L_{apri}$$

The scaling determines the weighting between observation and a-priori information.

LLR Arithmetic

ftw.

$$\sum_{j=1}^{J}{}^{\boxplus} L(u_j) = \log \frac{1 + \prod_{j=1}^{J} \tanh\left(L(u_j)/2\right)}{1 - \prod_{j=1}^{J} \tanh\left(L(u_j)/2\right)}$$

$$= 2\,\mathrm{artanh}\left(\prod_{j=1}^{J} \tanh\left(L(u_j)/2\right)\right)$$

variable U. The sign of $L_U(u)$ is the hard decision and the magnitude $|L_U(u)|$ is the reliability of this decision. Unless stated otherwise, the logarithm is the natural logarithm.

If the binary random variable U is conditioned on a different random variable or vector Y, then we have a conditioned log-likelihood ratio $L_{U|Y}(u|y)$ with

$$\begin{aligned} L_{U|Y}(u|y) &= \log \frac{P_U(u=+1|y)}{P_U(u=-1|y)} \\ &= \log \frac{P_U(u=+1)}{P_U(u=-1)} + \log \frac{p_{Y|U}(y|u=+1)}{p_{Y|U}(y|u=-1)} \\ &= L_U(u) + L_{Y|U}(y|u). \end{aligned} \quad (2)$$

When there is no danger of confusion, we will henceforth skip the indices for the probabilities and the log-likelihood ratios. Notice that the joint log-likelihood $L(u,y)$ is equal to the conditioned log-likelihood $L(u|y)$ since the probability $P(y)$ term can be canceled out. Using the relations

$$\begin{aligned} P(u_1 \oplus u_2 = +1) &= P(u_1=+1)\cdot P(u_2=+1) \\ &\quad + (1-P(u_1=+1))\cdot(1-P(u_2=+1)) \end{aligned} \quad (3)$$

with

$$P(u=+1) = \frac{e^{L(u)}}{1+e^{L(u)}} \quad (4)$$

it is easy to prove for statistically independent random variables U_1 and U_2

$$L(u_1 \oplus u_2) = \log \frac{1 + e^{L(u_1)} e^{L(u_2)}}{e^{L(u_1)} + e^{L(u_2)}} \quad (5)$$

$$\approx \mathrm{sign}(L(u_1)) \cdot \mathrm{sign}(L(u_2)) \cdot \min(|L(u_1)|, |L(u_2)|). \quad (6)$$

From now on we will use a special algebra for the log-likelihood ratio values $L(u)$: We use the symbol $\boxplus$ as the notation for the addition defined by

$$L(u_1) \boxplus L(u_2) \triangleq L(u_1 \oplus u_2) \quad (7)$$

with the additional rules

$$L(u) \boxplus \infty = L(u) \qquad L(u) \boxplus -\infty = -L(u) \quad (8)$$

and

$$L(u) \boxplus 0 = 0. \quad (9)$$

By induction one can further prove that

$$\sum_{j=1}^{J}{}^{\boxplus} L(u_j) \triangleq L\left(\sum_{j=1}^{J}{}^{\oplus} u_j\right) = \log \frac{\prod_{j=1}^{J}(e^{L(u_j)}+1) + \prod_{j=1}^{J}(e^{L(u_j)}-1)}{\prod_{j=1}^{J}(e^{L(u_j)}+1) - \prod_{j=1}^{J}(e^{L(u_j)}-1)} \quad (10)$$

is true. Using the relation $\tanh(u/2) = (e^u-1)/(e^u+1)$ we obtain [3]

$$\sum_{j=1}^{J}{}^{\boxplus} L(u_j) = \log \frac{1 + \prod_{j=1}^{J} \tanh(L(u_j)/2)}{1 - \prod_{j=1}^{J} \tanh(L(u_j)/2)} = 2\,\mathrm{artanh}\left(\prod_{j=1}^{J} \tanh(L(u_j)/2)\right) \quad (11)$$

and finally approximate it as in (6) by

$$\sum_{j=1}^{J}{}^{\boxplus} L(u_j) = L\left(\sum_{j=1}^{J}{}^{\oplus} u_j\right) \approx \left(\prod_{j=1}^{J} \mathrm{sign}(L(u_j))\right) \cdot \min_{j=1\cdots J} |L(u_j)|. \quad (12)$$

The reliability of the sum $\boxplus$ is therefore determined by the smallest reliability of the terms. From (11) we get the symmetrical relation

$$\tanh\left(\frac{1}{2}\sum_{j=1}^{J}{}^{\boxplus} L(u_j)\right) = \prod_{j=1}^{J} \tanh(L(u_j)/2) \quad (13)$$

to be used in the Appendix.

B. Soft Channel Outputs

Now, we will define more clearly what is meant by the "soft values" of a channel. If we encode the binary value u having a soft value $L(u)$ then we create coded bits x with soft values $L(x)$. For an (N,K)-systematic code, K of the bits x are equal to the information bits u. After transmission over a binary symmetric channel (BSC) or a Gaussian/fading channel we can calculate the log-likelihood ratio of x conditioned on the matched filter output y

$$\begin{aligned} L(x|y) &= \log \frac{P(x=+1|y)}{P(x=-1|y)} \\ &= \log\left(\frac{p(y|x=+1)}{p(y|x=-1)} \cdot \frac{P(x=+1)}{P(x=-1)}\right). \end{aligned} \quad (14)$$

With our notation we obtain

$$\begin{aligned} L(x|y) &= \log \frac{\exp(-\frac{E_s}{N_0}(y-a)^2)}{\exp(-\frac{E_s}{N_0}(y+a)^2)} + \log \frac{P(x=+1)}{P(x=-1)} \\ &= L_c \cdot y + L(x) \end{aligned} \quad (15)$$

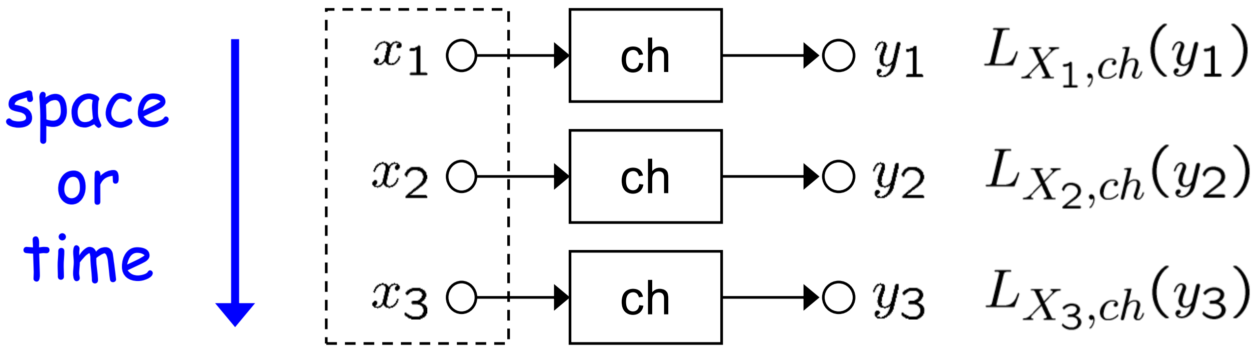
with $L_c = 4a \cdot E_s/N_0$. For a fading channel, a denotes the fading amplitude whereas for a Gaussian channel we set $a = 1$. For a BSC, L_c is the log-likelihood ratio of the crossover probabilities P_0, where $L_c = \log((1-P_0)/P_0)$. L_c is called the reliability value of the channel.

We further note that for statistically independent transmission, as in dual diversity or with a repetition code

$$L(x|y_1,y_2) = L_{c_1} y_1 + L_{c_2} y_2 + L(x). \quad (16)$$

Joachim Hagenauer (TU Munich)
A great advocate of LLR arithmetic.

Calculating with LLR



$$\mathsf{L}(X_1|\underline{Y}=\underline{y}) = \log \frac{P(X_1=+1|\underline{Y}=\underline{y})}{P(X_1=-1|\underline{Y}=\underline{y})} = \log \frac{P(X_1=+1)}{P(X_1=-1)} + \log \frac{P(\underline{Y}=\underline{y}|X_1=+1)}{P(\underline{Y}=\underline{y}|X_1=-1)}$$

$$= \underbrace{\log \frac{P(X_1=+1)}{P(X_1=-1)}}_{\text{a-priori L-value}} + \underbrace{\log \frac{P(Y_1=y_1|X_1=+1)}{P(Y_1=y_1|X_1=-1)}}_{\text{channel L-value}} + \underbrace{\log \frac{P(\underline{Y}_{[1]}=\underline{y}_{[1]}|X_1=+1)}{P(\underline{Y}_{[1]}=\underline{y}_{[1]}|X_1=-1)}}_{\text{extrinsic L-value}}$$

a-priori L-value | channel L-value | extrinsic L-value

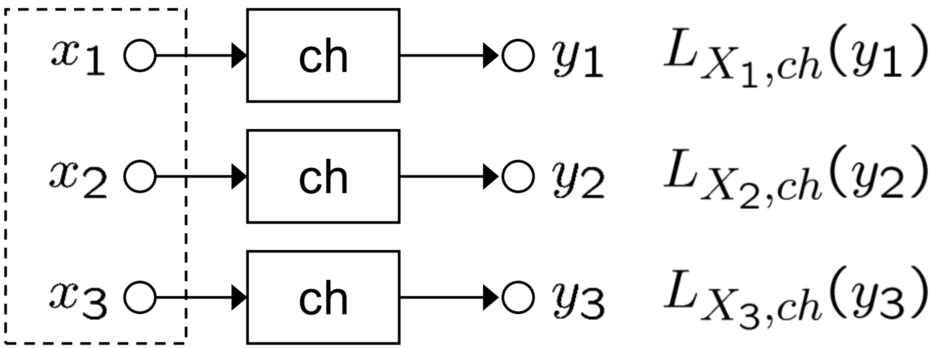
$$\mathsf{L}_{X_1,app}(\underline{y}) = \mathsf{L}_{X_1,apri} + \mathsf{L}_{X_1,ch}(y_1) + \mathsf{L}_{X_1,extr}(\underline{y}_{[1]})$$

Note on Notation

For a vector $\underline{y}$ with its i'th element removed we write

$$\underline{y}_{[i]} = \left[y_1, \cdots, y_{i-1}, y_{i+1}, \cdots, y_n\right]$$

Repeat Code



$$x_1 = x_2 = x_3$$

$$\mathsf{L}_{X_1,extr}(\underline{y}_{[1]}) = \log \frac{P(\underline{Y}_{[1]} = \underline{y}_{[1]} | X_1 = +1)}{P(\underline{Y}_{[1]} = \underline{y}_{[1]} | X_1 = -1)}$$

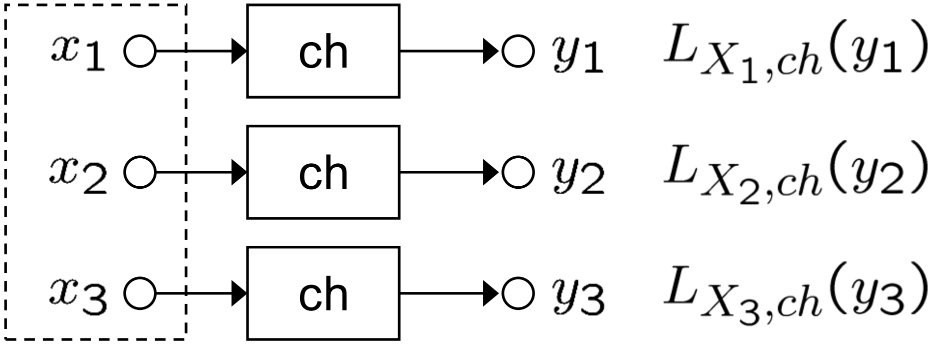
$$= \log \frac{P(Y_2 = y_2, Y_3 = y_3 | X_2 = +1, X_3 = +1)}{P(Y_2 = y_2, Y_3 = y_3 | X_2 = -1, X_3 = -1)}$$

$$= \log \frac{P(Y_2 = y_2 | X_2 = +1)}{P(Y_2 = y_2 | X_2 = -1)} + \log \frac{P(Y_3 = y_3 | X_3 = +1)}{P(Y_3 = y_3 | X_3 = -1)}$$

$$\mathsf{L}_{X_1,extr}(\underline{y}_{[1]}) = \mathsf{L}_{X_2,ch}(y_2) + \mathsf{L}_{X_3,ch}(y_3)$$

$$\mathsf{L}_{X_1,app}(\underline{y}) = \mathsf{L}_{X_1,apri} + \mathsf{L}_{X_1,ch}(y_1) + \mathsf{L}_{X_2,ch}(y_2) + \mathsf{L}_{X_3,ch}(y_3)$$

Single Parity-Check Code



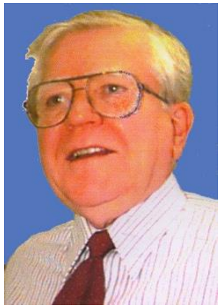
$$x_1 \cdot x_2 \cdot x_3 = +1$$

$$\mathsf{L}(X_1|\underline{Y}=\underline{y}) = \log\frac{P(X_1=+1)}{P(X_1=-1)} + \log\frac{P(Y_1=y_1|X_1=+1)}{P(Y_1=y_1|X_1=-1)} + \log\frac{P(\underline{Y}_{[1]}=\underline{y}_{[1]}|X_1=+1)}{P(\underline{Y}_{[1]}=\underline{y}_{[1]}|X_1=-1)}$$

$$= \log\frac{P(Y_1=y_1|X_1=+1)}{P(Y_1=y_1|X_1=-1)} + \log\frac{P(X_1=+1|\underline{Y}_{[1]}=\underline{y}_{[1]})}{P(X_1=-1|\underline{Y}_{[1]}=\underline{y}_{[1]})}$$

$$\mathsf{L}_{X_1,ch}(y_1) \qquad \mathsf{L}_{X_1,apri} + \mathsf{L}_{X_1,extr}(\underline{y}_{[1]})$$

Gallager-Lemma



Consider a sequence of m independent binary digits in which the i'th digit is 1 with probability P_i. Then the probability that an even number of digits are 1 is

$$\frac{1+\prod_{i=1}^{m}(1-2P_i)}{2}$$

The probability that an odd number of digits are 1 is

$$1-\frac{1+\prod_{i=1}^{m}(1-2P_i)}{2}=\frac{1-\prod_{i=1}^{m}(1-2P_i)}{2}$$

Single Parity-Check Code

$$\log \frac{P(X_1 = +1 | \underline{Y}_{[1]} = \underline{y}_{[1]})}{P(X_1 = -1 | \underline{Y}_{[1]} = \underline{y}_{[1]})}$$

$$P(X_1 = -1 | \underline{Y}_{[1]} = \underline{y}_{[1]}) = P(X_2 \cdot X_3 = -1 | \underline{Y}_{[1]} = \underline{y}_{[1]})$$

$$= \frac{1 - \prod_{i \neq 1} \left[1 - 2P(X_i = -1 | \underline{Y}_{[1]} = \underline{y}_{[1]})\right]}{2}$$

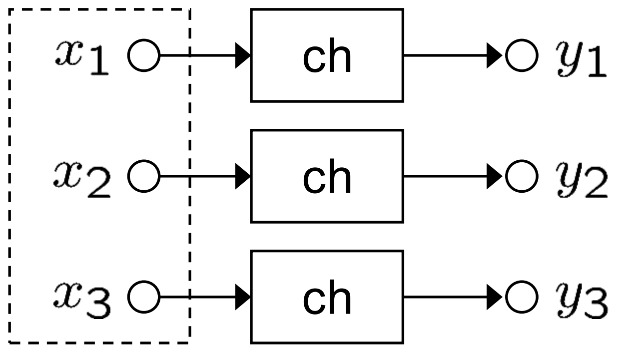
$$1 - 2P(X_1 = -1 | \underline{Y}_{[1]} = \underline{y}_{[1]}) = \prod_{i \neq 1} \left[1 - 2P(X_i = -1 | \underline{Y}_{[1]} = \underline{y}_{[1]})\right]$$

$$= \prod_{i \neq 1} [1 - 2P(X_i = -1 | Y_i = y_i)]$$

$$\tanh \frac{L_{X_1,apri} + L_{X_1,extr}(y_{[1]})}{2} = \prod_{i \neq 1} \tanh \frac{L_{X_i,intr}(y_i)}{2}$$

$$L_{X_1,apri} + L_{X_1,extr}(y_{[1]}) = 2 \cdot \tanh^{-1} \prod_{i \neq 1} \tanh \frac{L_{X_i,intr}(y_i)}{2}$$

Single Parity-Check Code



$L_{X_1,ch}(y_1)$

$L_{X_2,ch}(y_2)$

$L_{X_3,ch}(y_3)$

$$x_1 \cdot x_2 \cdot x_3 = +1$$

$$\mathsf{L}(X_1|\underline{Y}=\underline{y}) = L_{X_1,ch}(y_1) + 2\cdot\tanh^{-1}\left[\prod_{i\neq 1} \tanh \frac{L_{X_i,intr}(y_i)}{2}\right]$$

$$\mathsf{L}_{X_1,ch}(y_1) \quad \mathsf{L}_{X_1,apri} + \mathsf{L}_{X_1,extr}(\underline{y}_{[1]})$$

Calculating with LLR - Summary

If we receive independent noisy observations of the same binary random variable, we can add their channel L-values.

The L-value of a binary random variable that is the XOR of other random variables can be calculated using the tanh-rule.

Single Parity-Check Code Example

Parity-Check code x_1¢ x_2 ¢ x_3 = +1

Assumption: no a-priori information ($L_{ch} = L_{intr}$)

L_{intr}=[+1.50   -2.00   +2.00]

$$\mathrm{L}(X_1|\underline{Y}=\underline{y}) = L_{X_1,ch}(y_1)+2\cdot\tanh^{-1}\left[\prod_{i\neq 1}\tanh\frac{L_{X_i,intr}(y_i)}{2}\right]$$

L_{app}=[+0.18   -0.94   +0.94]

Decoder output: +1 -1 +1

This is not a codeword!!!

Single Parity-Check Code Sequence Probability

$$P(\underline{X}=\underline{x}|\underline{Y}=\underline{y}) = P(\underline{Y}=\underline{y}|\underline{X}=\underline{x}) \cdot \frac{P(\underline{X}=\underline{x})}{P(\underline{Y}=\underline{y})}$$

$$= \frac{P(\underline{X}=\underline{x})}{P(\underline{Y}=\underline{y})} \cdot \prod_i P(Y_i = y_i | X_i = x_i)$$

$$= \frac{P(\underline{X}=\underline{x})}{P(\underline{Y}=\underline{y})} \cdot \prod_i P(X_i = x_i | Y_i = y_i) \cdot \frac{P(Y_i = y_i)}{P(X_i = x_i)}$$

$$P(X=+1|Y=y) = \frac{e^{L_{ch}(y)}}{1+e^{L_{ch}(y)}} = \frac{e^{\frac{L_{ch}(y)}{2}}}{e^{-\frac{L_{ch}(y)}{2}} + e^{\frac{L_{ch}(y)}{2}}}$$

$$P(X=-1|Y=y) = \frac{1}{1+e^{L_{ch}(y)}} = \frac{e^{-\frac{L_{ch}(y)}{2}}}{e^{-\frac{L_{ch}(y)}{2}} + e^{\frac{L_{ch}(y)}{2}}}$$

$$\mathrm{argmax}_{\underline{x}\in\mathcal{C}} \left\{P(\underline{X}=\underline{x}|\underline{Y}=\underline{y})\right\} = \mathrm{argmax}_{\underline{x}\in\mathcal{C}} \left\{\sum_i L_{X_i,ch} \cdot X_i\right\}$$

Single Parity-Check Code
Sequence Probability - Example

L_{intr}=[+1.50 -2.00 +2.00]

$\underline{x}$	$\sum_i L_{X_i,intr} \cdot X_i$
+1 +1 +1	+1.50
+1 -1 -1	+1.50
-1 -1 +1	+2.50
-1 +1 -1	-5.50

Summary Decoding strategies

- If the decoder uses a-priori information it is called maximum a-posteriori (MAP) decoder.
- If the decoder uses no a-priori information it is called maximum likelihood (ML) decoder.
- The decoder can be designed to find the most likely symbol or the most likely sequence. This is independent of the use of a-priori information!