3F1 Signals and Systems: Handout 16

The Power Spectral Density for Continuous Time Signals

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Continuous-time random processes

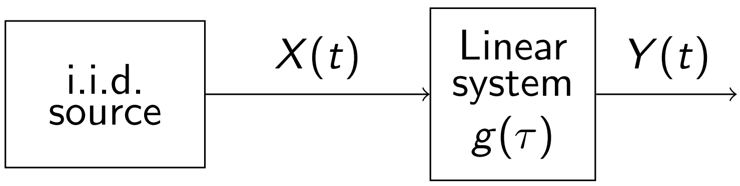
- we are now ready to tackle continuous-time random signals/processes
- we will begin by discussing **white noise**
- we will then repeat our derivation of **white noise through a linear system** as we did for discrete time
- as in discrete time, understanding the case of white noise through a linear system allows us to understand how any random process is affected by a linear system, because any random process can be modeled as the output of white noise through a linear system and feeding it through a further linear system is equivalent to cascading two linear systems

What White Noise is **NOT**

- in discrete time, white noise was simply an i.i.d. sequence of Gaussian random variables
- at every time k, the random variable X_k is Gaussian with mean 0 and variance σ^2, i.e., $X_k \sim \mathcal{N}(0, \sigma^2)$
- **how do we extend this concept to continuous time?**
- it would be tempting to consider a random function $X(t)$ where the variable $X(t)$ at any time t is an independent Gaussian random variable with zero mean and variance σ^2, i.e., $X(t) \sim \mathcal{N}(0, \sigma^2)$
- such a process is hard to imagine (any realisation/sample path would be discontinuous at every point) but that in itself doesn't make it wrong

The problem with i.i.d. Gaussian in continuous time

- consider putting such a random process through a linear system



- the output results from a convolution of the input with the filter response

$$Y(t) = X(t) * g(t) = \int_{-\infty}^{\infty} g(\tau)X(t-\tau)dt$$

- this is now a weighted average of infinitely many Gaussian random variables, weighted with the filter response $g(\tau)$
- by the Law of Large Numbers, $Y(t) = 0$ **for all times**

The problem with i.i.d. Gaussian in continuous time

- this isn't just a detail we can brush away: in the real world, white noise can **only** be observed through a linear filter
- any measurement device (microphone, photodiode, etc) is a linear filter
- **a continuous time i.i.d. Gaussian signal is unobservable**
- White Noise is **not** an i.i.d. Gaussian random process in continuous time
- so what is it?

So what is continuous time White Noise?

- White Noise is a bit of a **mystery!**
- textbooks spend long intricate chapters defining white noise as a limit of weighted pulses
- think of it as digital baseband modulation (Pulse Amplitude Modulation, as seen in 2P6 Communications)

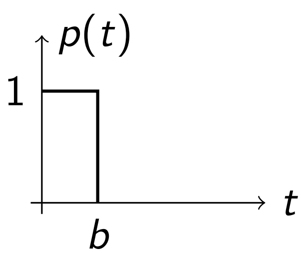
$$X(t) = \sum_{k=-\infty}^{\infty} X_k \, p_T(t - kT)$$

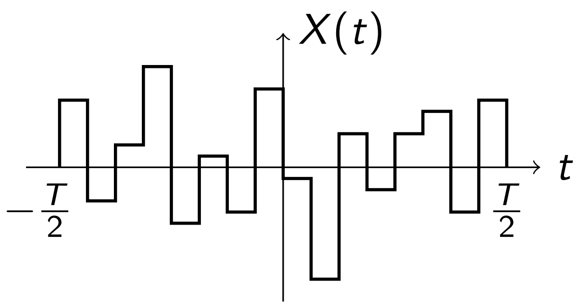
  with $X_k \sim \mathcal{N}(0, \sigma^2)$ and a pulse shape $p_T(t)$, **in the limit as $T \to 0$ and $\sigma^2 \to \infty$**
- this is most definitely a **"sloppy limit"** of the sort we saw in 1P4 EP7 Q5(ii). We don't know what it means for functions to approach a limit, and the limiting function is not well defined, very much like the Dirac impulse $\delta(t)$

Our approach

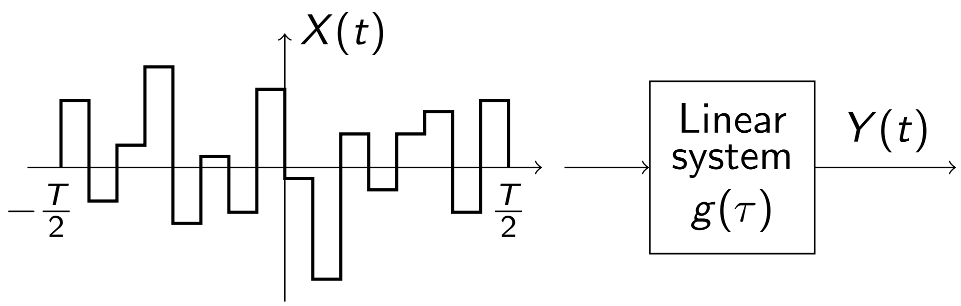
- as in 1P4 EP7 Q5(ii), we will use a rectangular pulse
- we use a rectangular pulse of height 1 and width b
- we construct a signal starting at $-T/2$ and ending at $T/2$, consisting of pulses modulated with independent Gaussian random variables $X_k \sim \mathcal{N}(0, 1/b)$ of variance $1/b$

$$X(t) = \sum_{k=\frac{-T}{2b}}^{\frac{T}{2b}-1} X_k p(t - kb)$$





Our approach (continued)



- we will analyse the expected squared magnitude of the Fourier spectrum of $X(t)$ and $Y(t)$ for a finite b and T
- note that we are now in continuous time so by "Fourier spectrum" we mean the regular Fourier transform **not** DTFT
- we will then see what happens when the width b of the pulse goes to zero, maintaining T constant
- recall that $X_k \sim \mathcal{N}(0, 1/b)$ so as b goes to zero the variance of the random variables defining $X(t)$ is going to infinity

Analysis

- Fourier transform of unit step $u(t)$ is $U(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}$ (Information data book)
- rectangle $p(t) = u(t) - u(t-b)$ is a difference of unit steps
- shift property: Fourier transform of $u(t-b)$ is $U(\omega)e^{-jb\omega}$
- hence
$$P(\omega) = \left(\pi\delta(\omega) + \frac{1}{j\omega}\right)(1 - e^{-jb\omega})$$
- recall that $X(t) = \sum_{k=\frac{-T}{2b}}^{\frac{T}{2b}-1} X_k p(t-kb)$
- shift property: Fourier transform of $p(t-kb)$ is $P(\omega)e^{-jkb\omega}$
- hence the Fourier transform of the input signal is

$$X(\omega) = \sum_{k=\frac{-T}{2b}}^{\frac{T}{2b}-1} X_k \left(\pi\delta(\omega) + \frac{1}{j\omega}\right)(1 - e^{-jb\omega})e^{-jkb\omega}$$

Analysis (continued)

- for $\omega \neq 0$, we can ignore the pulse function in the expression to obtain
$$X(\omega) = \sum_{k=-T/(2b)}^{T/(2b)-1} X_k \frac{1 - e^{-jb\omega}}{j\omega} e^{-jkb\omega}$$
- in discrete-time we defined $\mathcal{S}_{XX}(\theta) = \lim_{N\to\infty} \frac{1}{N} E[|X(\theta)|^2]$
- by analogy, we compute $\frac{1}{T}E[|X(\omega)|^2] = \frac{1}{T}E[X(\omega)X^*(\omega)]$

$$= \frac{1}{T}E\left[\left(\sum_{k=-\frac{T}{2b}}^{\frac{T}{2b}-1} X_k \frac{1 - e^{-jb\omega}}{j\omega} e^{-jkb\omega}\right)\left(\sum_{\ell=-\frac{T}{2b}}^{\frac{T}{2b}-1} X_\ell \frac{1 - e^{jb\omega}}{-j\omega} e^{j\ell b\omega}\right)\right]$$

$$= \frac{1}{T}\sum_{k=-T/(2b)}^{T/(2b)-1} \sum_{\ell=-T/(2b)}^{T/(2b)-1} E[X_k X_\ell] \frac{(1 - e^{-jb\omega})(1 - e^{jb\omega})}{\omega^2} e^{j(\ell-k)b\omega}$$

$$= \frac{1}{T}\sum_{k=-T/(2b)}^{T/(2b)-1} E[X_k^2] \frac{2 - 2\cos(b\omega)}{\omega^2} \text{ since } E[X_k X_\ell] = 0 \text{ for } k \neq \ell$$

Analysis (continued)

- we obtained $\frac{1}{T}E[|X(\omega)|^2] = \frac{1}{T}\sum_{k=-T/(2b)}^{T/(2b)-1} E[X_k^2]\frac{2-2\cos(b\omega)}{\omega^2}$
- recall that $E[X_k^2] = 1/b$
- the expression inside the sum does no depend on k and there are T/b values for k
- hence we have

$$\frac{1}{T}E[|X(\omega)|^2] = \frac{1}{T}\frac{T}{b}\frac{1}{b}\frac{2-2\cos(b\omega)}{\omega^2} = \frac{1}{b^2}\frac{2-2\cos(b\omega)}{\omega^2}$$

- we now take the limit as $b \to 0$ to obtain

$$\lim_{b\to 0}\frac{2-2\cos(b\omega)}{b^2\omega^2} = \lim_{b\to 0}\frac{2-2\left(1-\frac{b^2\omega^2}{2}+O(b^4)\right)}{b^2\omega^2} = 1$$

- we have shown that, for $\omega \neq 0$,

$$\mathcal{S}_{XX}(\omega) = \lim_{b\to 0}\frac{1}{T}E[|X(\omega)|^2] = 1$$

  which is the same result we obtained in discrete time

Summary

- White Noise, though hard to define in continuous time, has a constant power spectrum density $\mathcal{S}_{XX}(\omega) = 1$ for all $\omega \neq 0$
- I'm not sure what happens at $\omega = 0$ (and it doesn't matter?)
- we can use the same trick as in discrete time to conclude that

$$\begin{aligned}\mathcal{S}_{YY}(\omega) &= \lim_{b\to 0}\frac{1}{T}E[|Y(\omega)|^2] \\ &= \lim_{b\to 0}\frac{1}{T}E[|G(\omega)X(\omega)|^2] \\ &= |G(\omega)|^2\mathcal{S}_{XX}(\omega) = |G(\omega)|^2\end{aligned}$$

- as in discrete time, it is easy to show using the convolution property of Fourier transforms that for any continuous stationary ergodic random process $X(t)$, the PSD $\mathcal{S}_{XX}(\omega)$ is the Fourier transform of the auto-correlation function

$$r_{XX}(\tau) = E[X(t)X(t+\tau)]$$

- all other properties regarding CSD $\mathcal{S}_{XY}(\omega)$ etc. remain