3F1 Signals and Systems: Handout 14

Random processes - an introduction

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Motivation

- the last 3 lectures of 3F1 are about (mainly) **continuous time random processes**
- but wait, I hear you say...
- isn't 3F1 all about **deterministic discrete-time** signals and systems?
- nothing in 3F1 so far has been about probability and (almost) nothing about continuous time
- you are right...

Course history

- Pre-2016, many info modules were pieced together from small often unrelated parts: 3F1 was control theory, information theory and random processes
- The "David MacKay reform" in 2016 introduced a thematic structure and created new fantastic modules focused on one essential topic, such as "3F7 Information Theory" and "3F8 Inference"
- Discrete time random processes went to 3F3.
- Continuous time random processes were kept in 3F1 so that students who wanted to specialise in control theory could learn about random processes without having to take 3F3.
- As it turns out, we will cover a combination of discrete and continuous time random processes using an approach that differs from 3F3 but the outcome is the same: this part of the course (and in particular the examples paper) may well seem familiar to those taking 3F3.

What we will learn...

For stationary random processes,

		Discrete time	Continuous time
Time	Auto/cross correlation	$r_{XX}[k] = E[X_\ell X_{\ell+k}]$ $r_{XY}[k] = E[X_\ell Y_{\ell+k}]$	$r_{XX}(\tau) = E[X(t)X(t+\tau)]$ $r_{XY}(\tau) = E[X(t)Y(t+\tau)]$
Frequency	PSD	$\mathcal{S}_{XX}(\theta), \mathcal{S}_{XY}(\theta)$	$\mathcal{S}_{XX}(\omega), \mathcal{S}_{XY}(\omega)$
Linear filtering		$\begin{cases} \mathcal{S}_{YY}(\theta) = \lvert H(\theta)\rvert^2 \mathcal{S}_{XX}(\theta) \\ \mathcal{S}_{XY}(\theta) = H(\theta)\mathcal{S}_{XX}(\theta) \end{cases}$	$\begin{cases} \mathcal{S}_{YY}(\omega) = \lvert H(\omega)\rvert^2 \mathcal{S}_{XX}(\omega) \\ \mathcal{S}_{XY}(\omega) = H(\omega)\mathcal{S}_{XX}(\omega) \end{cases}$

- PSD: power spectral density
- most of the content of these 3 last lectures aims to motivate and explain the content of this slide

What is a random process?

- We have learned about signals:

Discrete time	Continuous time
A sequence $\{u_k\}$ of values indexed by the integer k	A function $f(t)$ that assigns values for every real valued t

- Random processes are **random signals**
- Wait... Are we equipped to deal with random sequences and random functions?
- In 2P7, we learned about random variables and vectors:

Random variable	Random vector
X, pdf $f_X(x)$ such that $\int_{-\infty}^{\infty} f_X(x)dx = 1$	(X, Y), joint pdf $f_{XY}(x, y)$ s. t. $\int_{-\infty}^{\infty} f_{XY}(x, y)dy = f_X(x)$

- We can deal with a vector of any finite size n, $\underline{X} = (X_1, X_2, \ldots, X_n)$ but can we deal with an "infinite vector" $\{X_1, X_2, \ldots\}$ (a sequence) or with an infinitely infinite vector $\{X_t\}_{t \in \mathbb{R}}$ (a function)? **No!**

Random processes and probability theory

- we cannot define a joint probability density function for all the elements in a sequence or all the values of a function!
- **a random process is a random signal (discrete or continuous) for which the joint density of any collection of any n signal values $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$ is known and well defined**
- expectations $E[f(X_{t_1}, X_{t_2}, \ldots, X_{t_n})]$ and other derived probabilistic quantities (e.g. entropy) are well defined
- in some cases, it will be necessary to take limits of derived quantities as n goes to infinity
- as engineers, our use of limits is sometimes **"sloppy"**. What do I mean by this?

Engineering and "sloppy" limits

- are all the limits we've been taking "sloppy" limits? No...
- here is an example of a "precise" limit:

$$\lim_{k\to\infty} \frac{k+1}{2k-1} = \frac{1}{2}$$

- The definition of a limit in this case is that, for every $\epsilon > 0$, there exists an index k_0 for which

$$\left|\frac{k+1}{2k-1} - \frac{1}{2}\right| < \epsilon \text{ for all } k \geq k_0$$

- in other words the sequence becomes arbitrarily close to the limit for large indices.
- this is a precise definition of "limit" and there is nothing wrong with it
- the next slide will show an example of a "sloppy" limit we've taken in our tripos.

1P4 Maths Examples Paper 7, Question 5

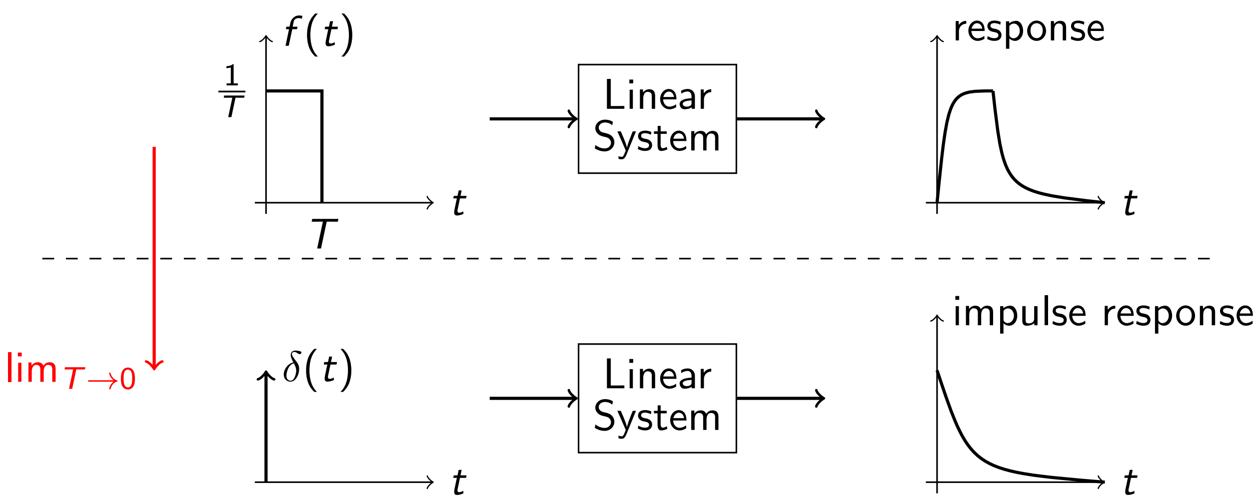
5. For a certain system, the output ( ) to an input ( ) satisfies the differential equation

$$— + \quad = (\ )$$

(i) Find the step response of the system, by solving the equation with $(\ ) = \begin{matrix} 0 & < 0 \\ 1 & \geq 0 \end{matrix}$ and $= 0$ for $\quad 0$.

(ii) Hence write down the output of the system to input $(\ ) = \begin{matrix} 0 & & < 0 \\ 1 & 0 & \\ 0 & & > \end{matrix}$, with $= 0$ for $\quad 0$. Verify that this becomes the impulse response in the limit $\to 0$.

The "sloppy" premise for Question 5(ii)



- "As $T \to 0$, the input $f(t)$ tends towards an impulse so the output must tend towards the system impulse response"
- What does it mean for a function to tend towards a function, particularly one as bizarre as a Dirac impulse?

"Sloppy" limits

- mathematics students study function limits for 3 years before they learn about random processes or Fourier analysis
- as engineers, we are happy to address some limits on a purely intuitive basis
- we should bear in mind that some such limits can be delicate
- we will need to take delicate limits to understand random processes and there will be some parallels with the way we used limits in 1P4 examples paper 7

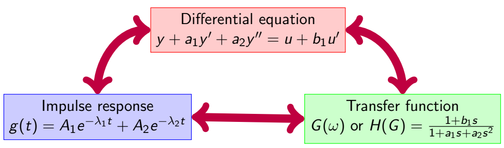
What do we want from random processes?

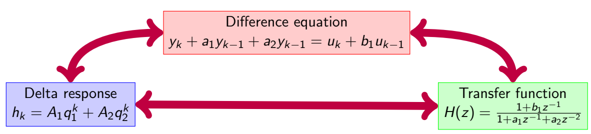
- random processes are random signals
- like in the rest of 3F1 and 1P4/2P6, we want to understand what happens when signals are fed through linear systems
- a linear system is a linear Ordinary Differential Equation (ODE) or a linear Ordinary Difference Equation (in discrete time) with constant coefficients
- when applied to random signals (random processes) linear systems equations are known as **Stochastic Differential (difference) Equations (SDE) with constant coefficients**

	Continuous time	Discrete time
Deterministic	Linear **ODE** with constant coeffients	Linear **Ordinary Difference Equation** with constant coefficients
Random	Linear **SDE** with constant coefficients	Linear **Stochastic Difference Equation** with constant coefficients

Responses and transfer functions

- Remember the tools we have acquired so far:





- do they still work with random signals? **No!**
- **The system is unchanged.** It still has a response and a transfer function in the Laplace/Fourier/z/DTFT domain
- However, **the signals are "unknown"**. We cannot simply take their Laplace/Fourier/z transform/DTFT
- the base of the triangle (response, transfer function) has become disconnected from the top of the triangle (differential/difference equation)
- We can not write $Y(\omega) = H(\omega)X(\omega)$ or $Y(\theta) = H(\theta)X(\theta)$ because the input signal no longer has a known transform $X(\omega)$ or $X(\theta)$.

An idea: take expectations of the system equation...

- without loss of generality, consider a discrete-time signal such that $E[X_k] = 0$ for all k.
- let's consider for example putting this zero-mean signal through a difference equation

$$Y_k + a_1 Y_{k-1} = b_0 X_k + b_1 X_{k-1}$$

- we take the expectation of the difference equation and obtain

$$E[Y_k] + a_1 E[Y_{k-1}] = b_0 E[X_k] + b_1 E[X_{k-1}] = b_0 \cdot 0 + b_1 \cdot 0 = 0$$

  in other words

$$E[Y_k] = -a_1 E[Y_{k-1}]$$

  which says that the mean of the output signal $\{Y_k\}$ is a geometric sequence. If the system is stable, this is a transient response. We can say nothing about the stationary response of a system to a stochastic input by analysing the mean.

Time domain expectations, further comments

- we've shown that taking expectations of a difference equation with a zero-mean input is not useful
- Why was there no loss of generality in assuming zero mean? Because if $E[X_k] \neq 0$, we can separate X into a deterministic signal $\hat{x}_k = E[X_k]$ and a zero-mean signal $\tilde{X}_k = X_k - \hat{x}_k$

$$X_k = E[X_k] + (X_k - E[X_k]) = \hat{x}_k + \tilde{X}_k$$

  where $E[\tilde{X}_k] = 0$. By linearity, the response of the system is the deterministic response to $\{\hat{x}_k\}$ plus the response to a zero-mean signal.

Another idea: can we take expectations of a transform?

- What about taking expectations of a transform?
- Consider taking the DTFT of the difference equation:

$$\sum_{k=-\infty}^{\infty} (Y_k + a_1 Y_{k-1}) e^{-j\theta k} = \sum_{k=-\infty}^{\infty} (b_0 X_k + b_1 X_{k-1}) e^{-j\theta k}$$

- Good news: there appears to be nothing wrong with the "shift" and other properties of the DTFT when applied to stochastic signals... We can re-write as

$$(1 + a_1 e^{-j\theta}) \sum_{k=-\infty}^{\infty} Y_k e^{-j\theta k} = (b_0 + b_1 e^{-j\theta}) \sum_{k=-\infty}^{\infty} X_k e^{-j\theta k}$$

- Can we take expectations? **No!**

The problem with expectations of a transform

- taking an expectation results in an expression

$$E\left[\sum_{k=-\infty}^{\infty} Y_k e^{-j\theta k}\right]$$

- What do we mean by this?
- We are tempted to write

$$E\left[\sum_{k=-\infty}^{\infty} Y_k e^{-j\theta k}\right] = \int_{\underline{y}} f_{\underline{Y}}(\underline{y}) \left(\sum_{k=-\infty}^{\infty} Y_k e^{-j\theta k}\right) d\underline{y}$$

- this is **wrong!** What we denoted as $\underline{Y}$ is not a vector but an infinite signal and there is no such thing as the joint pdf of all components of the sequence (as we noted earlier...)

Remedy for expectations of a transform

- We need to take some sort of limit, for example by writing

$$\text{DTFT}(Y) = \lim_{N\to\infty} \sum_{k=-N/2}^{N/2} Y_k e^{-j\theta k}$$

- Now take expectations

$$E\left[\text{DTFT}(Y)\right] = E\left[\lim_{N\to\infty} \sum_{k=-N/2}^{N/2} Y_k e^{-j\theta k}\right] \stackrel{?}{=} \lim_{N\to\infty} E\left[\sum_{k=-N/2}^{N/2} Y_k e^{-j\theta k}\right]$$

- **Is this legal? Not in general. . .**
- Exchanging limits and expectations is only possible in rare cases. There are advanced theorems beyond the scope of the Engineering tripos governing this (e.g. Lebesque's Dominated Convergence Theorem).
- in this course we will avoid taking limits and work with finite N where possible, and sometimes when we have no choice we will resign ourselves to taking "sloppy" limits

Expectations of transforms: conclusion and further steps

- applying what we learned so far, we see that for a zero-mean input $E[X_k] = 0$, using a "sloppy" limit,

$$E\left[\text{DTFT}(X)\right] = \lim_{N\to\infty} E\left[\sum_{k=-N/2}^{N/2} X_k e^{-j\theta k}\right]$$
$$= \lim_{N\to\infty} \sum_{k=-N/2}^{N/2} E[X_k] e^{-j\theta k} = 0$$

- so far we've only succeeded in showing that the expected DTFT of the input and output in our linear difference equation is zero.
- this is in line with the analysis we did in the time domain
- next lecture, we will try another idea: take the expectation of the **magnitude squared** of the DTFT.