3F1 Signals and Systems: Handout 10

Digital Filtering: Part II (FIR design and examples)

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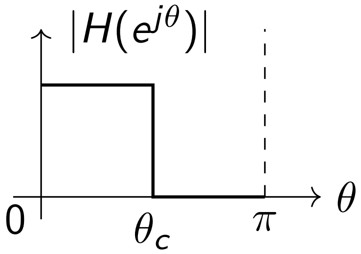
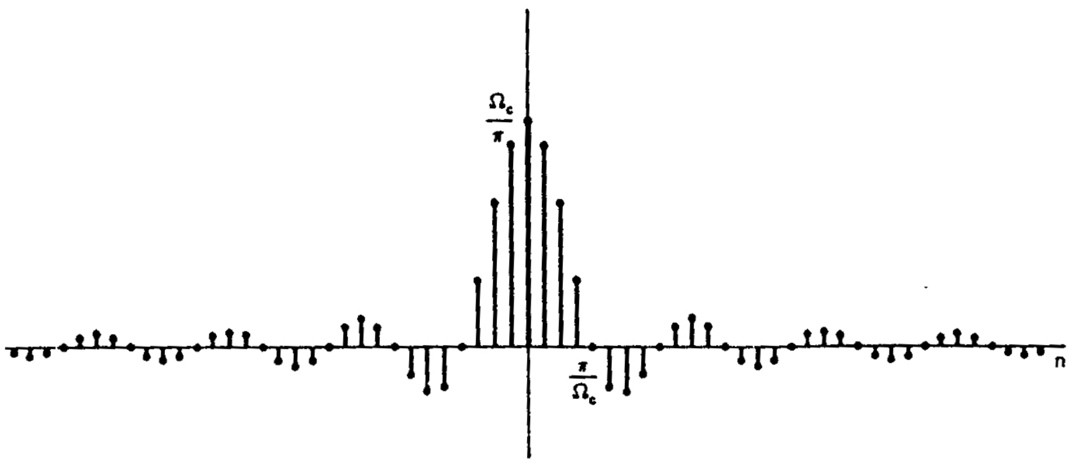
Michaelmas Term 2025

Recall: ideal lowpass delta response



$$h_k = \frac{1}{2\pi}\int_{-\theta_c}^{\theta_c} e^{j\theta k}\,d\theta = \frac{\theta_c}{\pi}\,\mathrm{sinc}(\theta_c k)$$



- It is non-causal and has a two-sided infinite delta response.
- Not realisable as a real-time linear system $\longrightarrow$ approximate!

FIR filter design

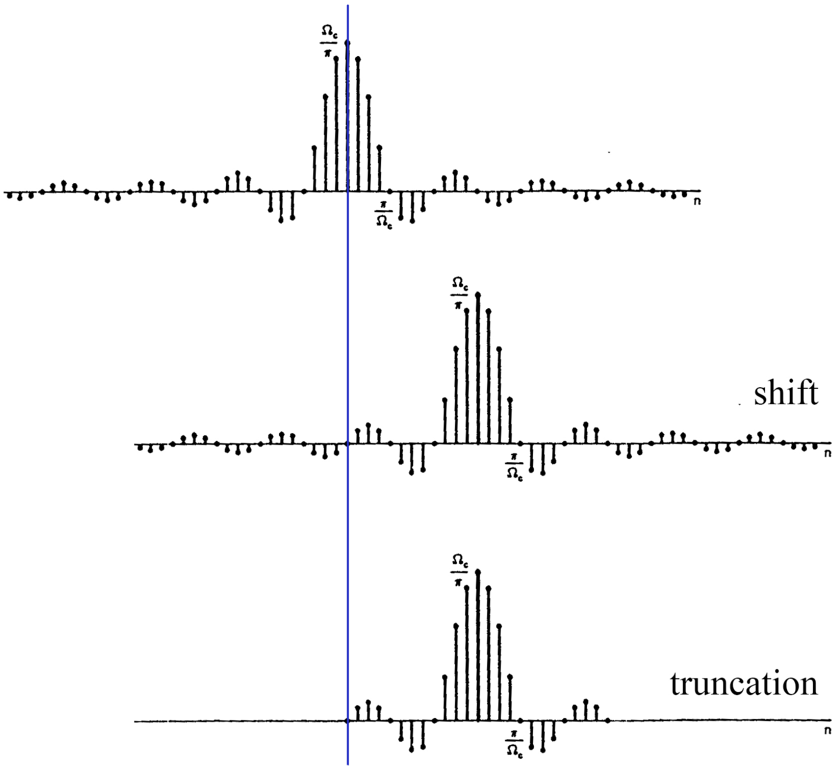
h_k is the non-causal impulse response of the ideal filter

New (finite impulse response) filter G derived by

- shift, and
- truncation at $N+1$ samples

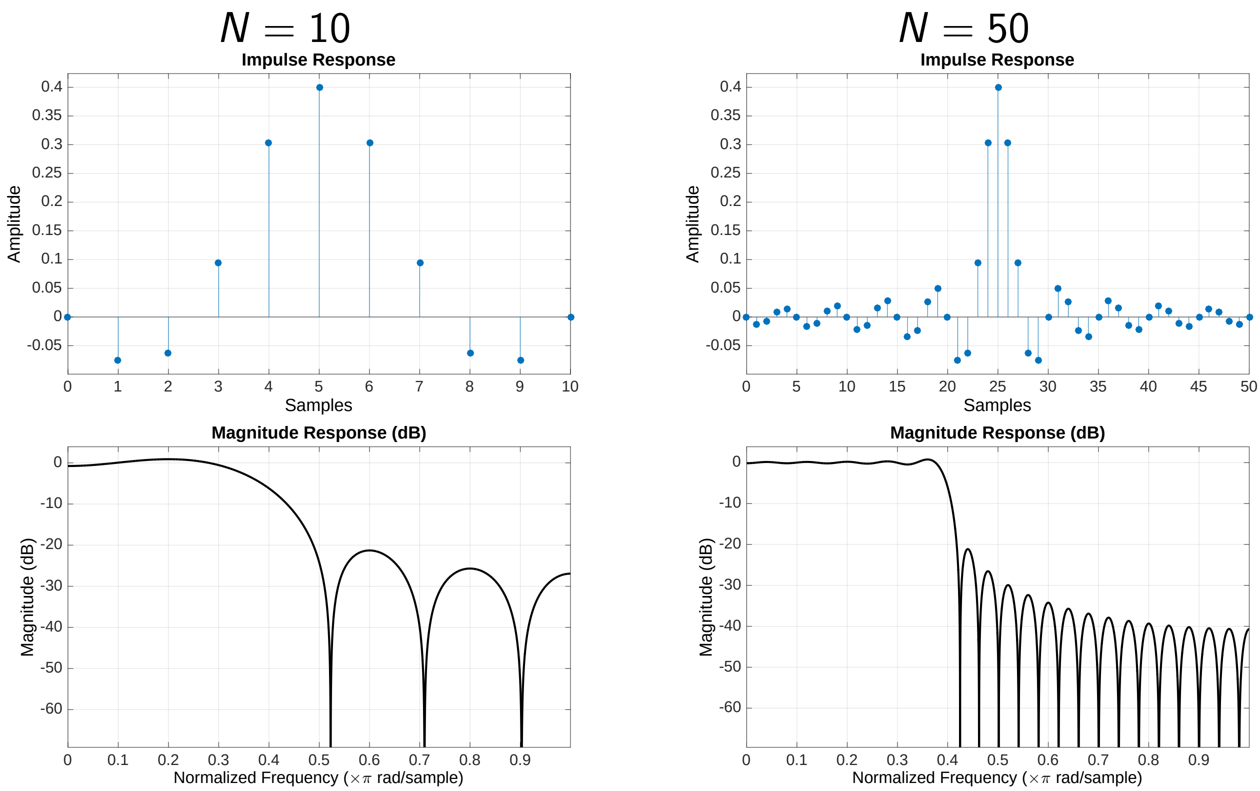
of the ideal response

Intuition: truncation of "small" samples has modest impact...



$$\text{FIR: } G(z) = \sum_{k=0}^{N} g_k z^{-k} \qquad g_k = h_{k-N/2} \quad \to N+1 \text{ samples!}$$

Truncated ideal filters



- Larger N → smaller samples discarded → reduced distortion, **but** longer filters, larger delay, bad stopband attenuation
- To understand truncation, we revisit properties of the DTFT

The DTFT's reverse convolution property

- In Examples Paper 1, Question 1(b), you proved that Fourier Series satisfy the periodic convolution property:

$$h(t) = \frac{1}{T}\int_{-T/2}^{T/2} g(\tau)f(t-\tau)d\tau \longrightarrow h_n = g_n f_n$$

- We have seen in Lecture 3 that Fourier series $\Longleftrightarrow$ DTFT with time and frequency swapped (and $T = 2\pi$)
- Hence for the DTFT

$$w_n = u_n v_n \longrightarrow W(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} U(\varphi)V(\theta-\varphi)d\varphi$$

- Multiplication in the time-domain results in periodic convolution the DTFT spectra in the frequency-domain
- The DTFT has 2 convolution properties (from k to θ):
  1. **discrete convolution** in time $\longrightarrow$ **multiplication** in frequency
  2. **multiplication** in time $\longrightarrow$ **periodic convolution** in frequency

The effect of truncation

Shifted desired impulse response

$$\tilde{h}_k = h_{k-N/2}$$

Truncated unit sample response

$$g_k = h_k \qquad 0 \le k \le N$$

Truncation = multiplication by a rectangular window

$$g_k = h_k w_k \qquad\qquad w_k = \begin{cases} 1 & 0 \le k \le N \\ 0 & \text{otherwise} \end{cases}$$

Equivalent to periodic convolution in the frequency domain:

$$G(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\varphi)W(\theta-\varphi)d\varphi$$

DTFT of the rectangular window

$$w_k = \begin{cases} 1 & 0 \le k \le N \\ 0 & \text{otherwise} \end{cases}$$

$$W(\theta) = \sum_{k=-\infty}^{\infty} w_k e^{-j\theta k} = \sum_{k=0}^{N} e^{-j\theta k}$$

$$= \frac{1 - e^{-j\theta(N+1)}}{1 - e^{-j\theta}} = \frac{e^{-j\theta(N+1)/2}\left(e^{j\theta(N+1)/2} - e^{-j\theta(N+1)/2}\right)}{e^{-j\theta/2}\left(e^{j\theta/2} - e^{-j\theta/2}\right)}$$

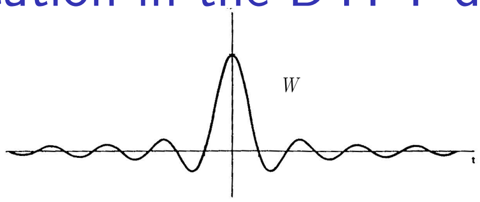
$$= e^{-j\theta N/2} \frac{\sin\left(\frac{\theta(N+1)}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$$

- if N reasonably large, $\theta\frac{N+1}{2} \gg \theta/2$ so the denominator is $\sin(\theta/2) \approx \theta/2$ and hence

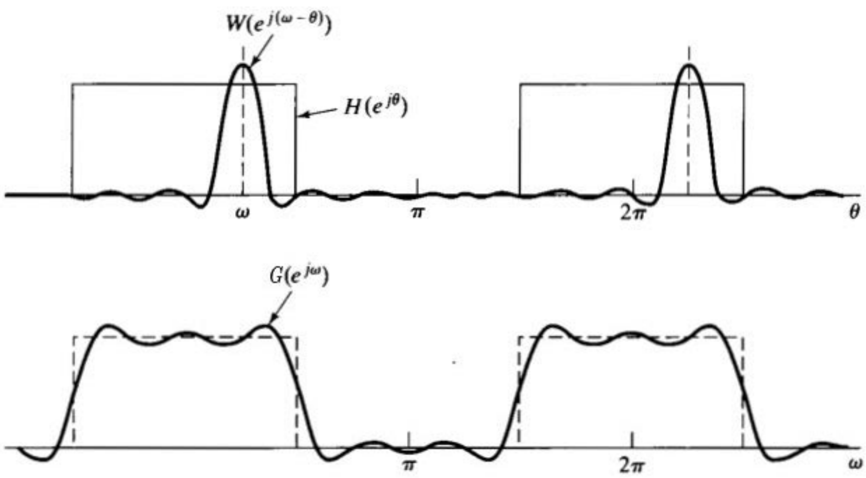
$$|W(\theta)| \approx (N+1)\left|\text{sinc}\left(\frac{\theta(N+1)}{2}\right)\right|$$

- This is approximately a sinc function.

The effect of truncation in the DTFT domain



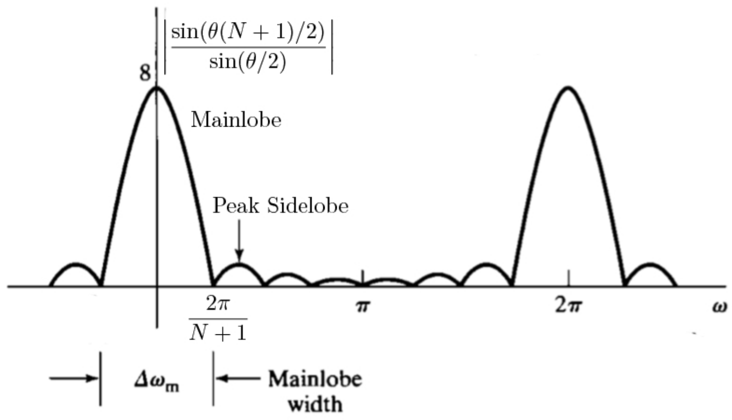
$G = H * W$

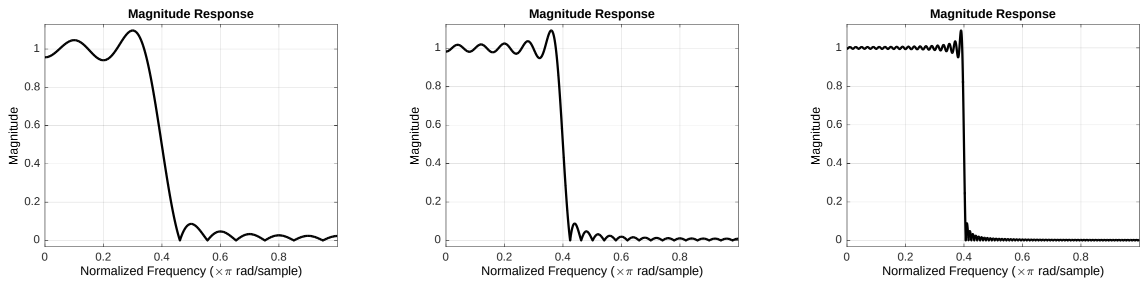


The illustration uses regular convolution rather than periodic convolution, but assuming that the sinc is narrow, the effect of periodicity should be small.

Analysing the effect of truncation

$|W(e^{j\theta})|$

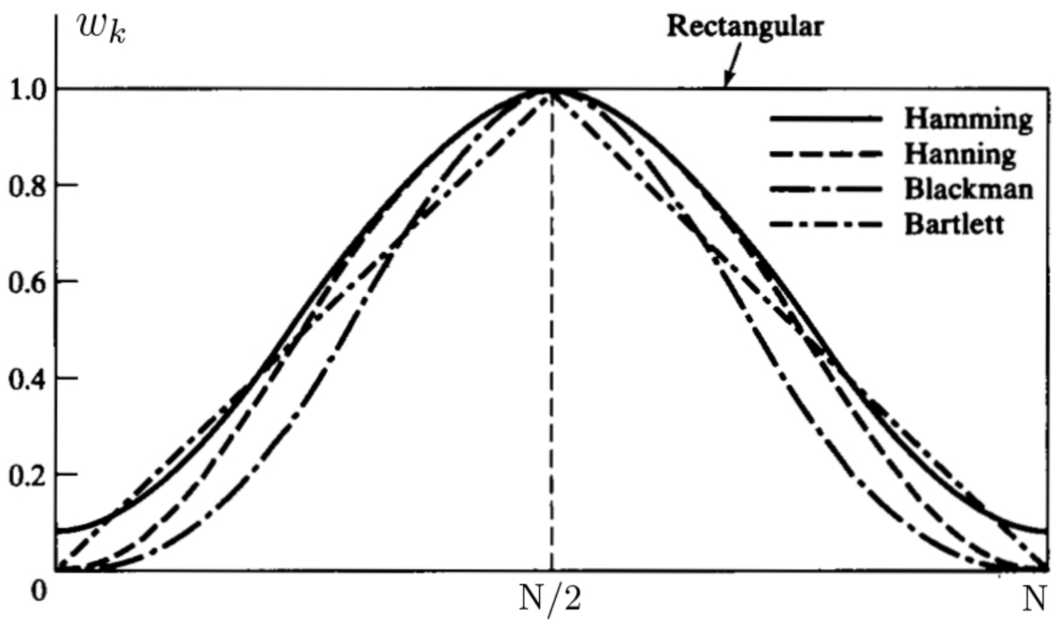




- Transition band is related to main lobe. It reduces as $N \to \infty$.
- Ripples of G are related to the area under sidelobes, which remains constant as N increases.

How to improve?

Design by window method



$$g_k = h_k w_k$$

Reduce frequency distortion by adopting different windows

Windows

Rectangular

$$w_k = \begin{cases} 1 & 0 \leq k \leq N \\ 0 & \text{otherwise} \end{cases}$$

Bartlett (triangular)

$$w_k = \begin{cases} 2k/N & 0 \leq k \leq N/2 \\ 2 - 2k/N & M/2 < k \leq N \\ 0 & \text{otherwise} \end{cases}$$

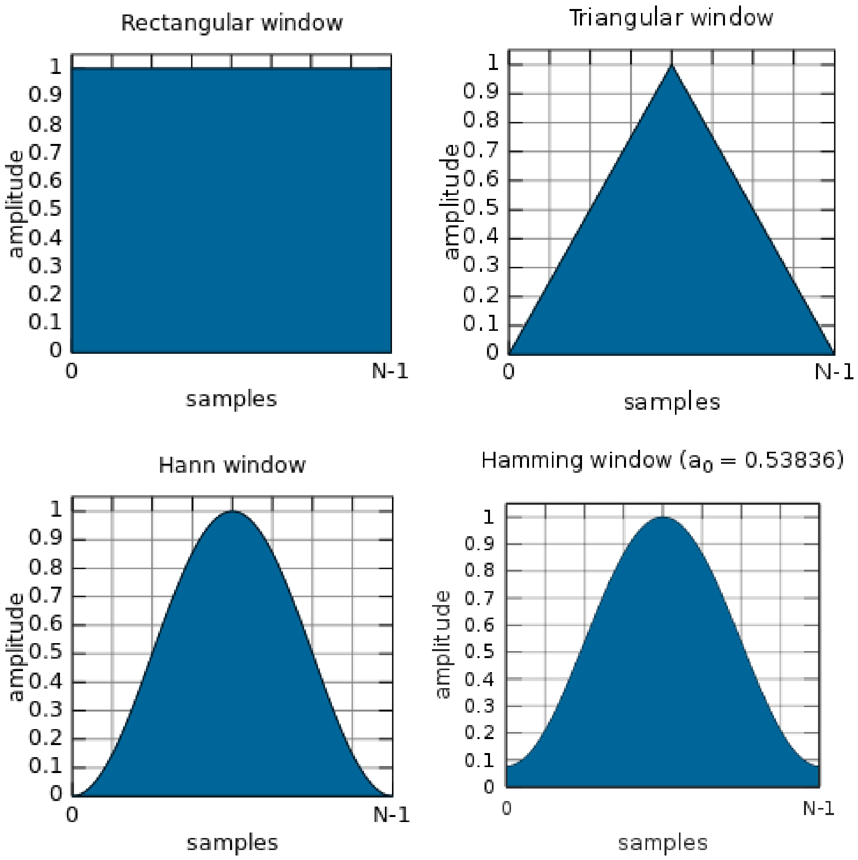
Hanning

$$w_k = \begin{cases} 0.5 - 0.5\cos(2\pi k/N) & 0 \leq k \leq N \\ 0 & \text{otherwise} \end{cases}$$

Hamming

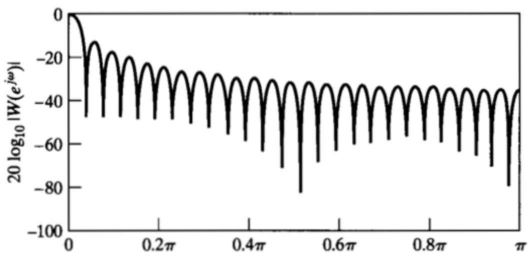
$$w_k = \begin{cases} 0.54 - 0.46\cos(2\pi k/N) & 0 \leq k \leq N \\ 0 & \text{otherwise} \end{cases}$$

Window plots

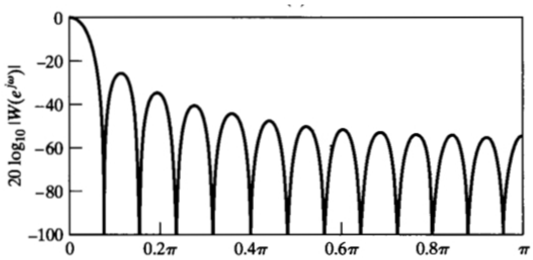


From `https://en.wikipedia.org/wiki/Window_function`

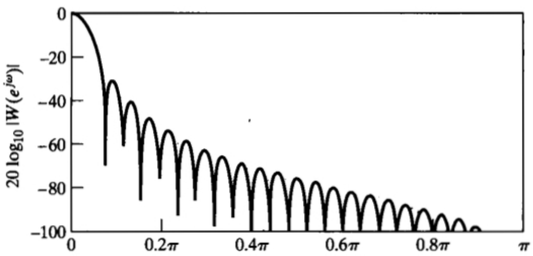
Windows in the frequency domain



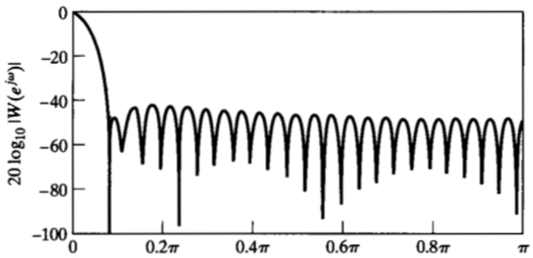
Rectangular



Triangular



Hanning



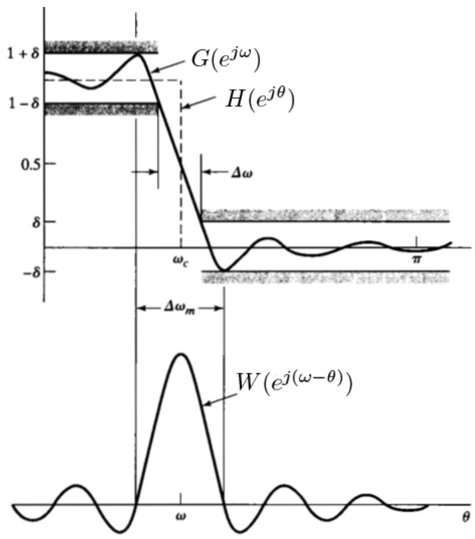
Hamming

Window characteristics

$$G(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\theta})W(e^{j(\omega-\theta)})d\theta$$

COMPARISON OF COMMONLY USED WINDOWS

Type of Window	Peak Side-Lobe Amplitude (Relative)	Approximate Width of Main Lobe	Peak Approximation Error, $20\log_{10}\delta$ (dB)
Rectangular	−13	$4\pi/(N+1)$	−21
Bartlett	−25	$8\pi/N$	−25
Hanning	−31	$8\pi/N$	−44
Hamming	−41	$8\pi/N$	−53
Blackman	−57	$12\pi/N$	−74



Shape and duration of the window to control the resulting filter properties.

- Smaller side lobes yield better approximations of the ideal response.
- Narrower transition bandwidth for increasing N. Symmetric at cutoff ω_c.
- Same δ for passband error and stopband approximation

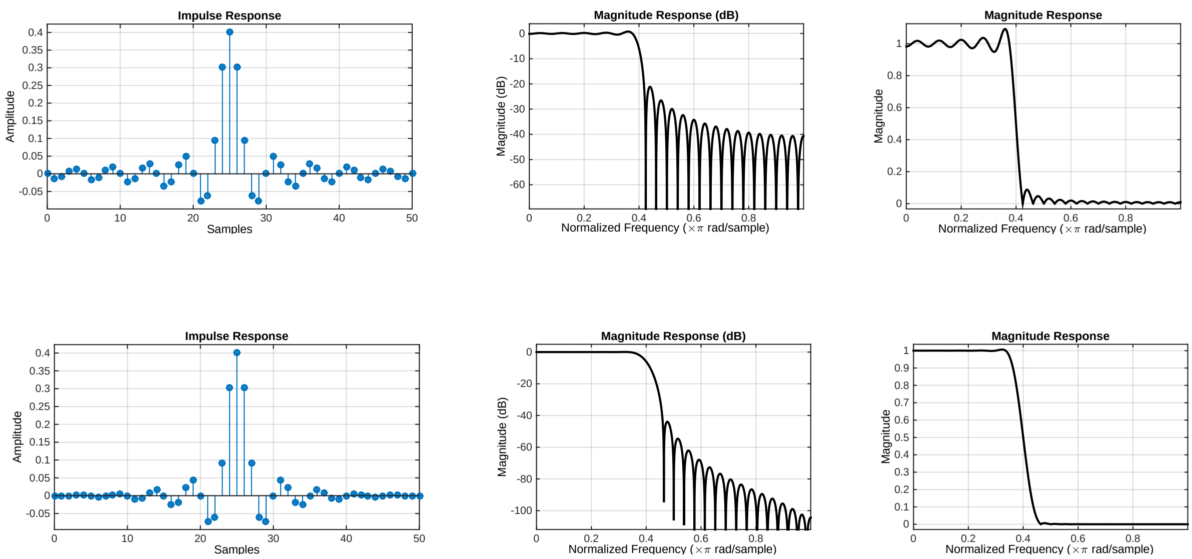
Window method

The window method is conceptually simple and can quickly design filters to approximate a given target response. It does not explicitly impose amplitude response constraints, such as passband ripple or stopband attenuation, so it has to be used iteratively to produce designs which meet such specifications.

1. Select a suitable window function w_k.
2. Specify an ideal frequency response H.
3. Compute the coefficients of the ideal filter h_k.
4. Multiply the ideal coefficients by the window function to give the filter coefficients and delay to make causal.
5. Evaluate the frequency response of the resulting filter and iterate 1-5 if necessary.

Example: Rectangular vs Hanning, N=50

g_k $\qquad$ $W(\theta)$ $\qquad$ $G(\theta)$



Example: Design of a low pass filter (Steps 1 and 2)

Passband $\theta_p = 0.2\pi$
Stopband $\theta_s = 0.3\pi$
Approximation error $\delta = 0.01$

COMPARISON OF COMMONLY USED WINDOWS

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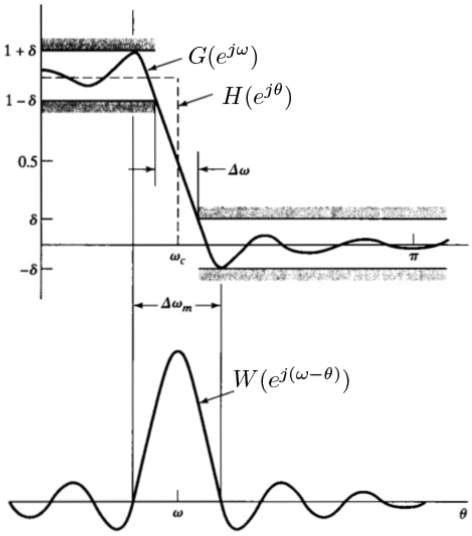


Approximation error: -40 dB $\rightarrow$ Hanning window.
Cutoff frequency: $\theta_c = \frac{\theta_s+\theta_p}{2}\pi = 0.25\pi$.
Mainlobe width: $\theta_s - \theta_p = 0.1\pi \rightarrow N = 80$.

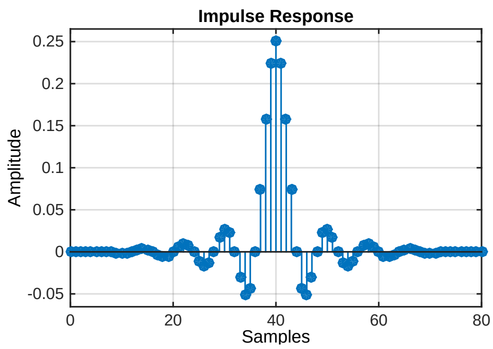
Example: Design of a low pass filter (Steps 3 and 4)

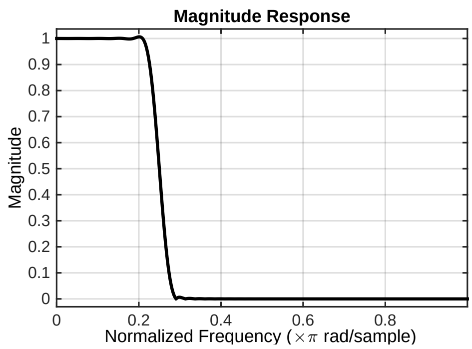
Ideal filter by inverse Fourier transform $\mathcal{F}^{-1}$

$$h_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\theta})e^{j\theta k}d\theta = \frac{1}{2\pi}\int_{-\theta_c}^{\theta_c} e^{j\theta k}d\theta = \frac{\sin(k\theta_c)}{\pi k} = \frac{\theta_c}{\pi}\mathrm{sinc}(k\theta_c)$$

Delay and product by window

$$g_k = h_{k-N/2}w_k$$





Multi-band FIR design

Design highpass, bandpass,...? **Composition of lowpass filters**

The ideal bandpass filter

$$H(e^{j\theta}) = \begin{cases} 1 & \theta_1 \leq \theta \leq \theta_2 \\ 0 & \text{otherwise} \end{cases}$$

$$= \mathrm{Lowpass}_{[0,\theta_2]}(e^{j\theta}) - \mathrm{Lowpass}_{[0,\theta_1]}(e^{j\theta})$$

where

$$\mathrm{Lowpass}_{[0,\theta_c]}(e^{j\theta}) = \begin{cases} 1 & 0 \leq \theta \leq \theta_c \\ 0 & \text{otherwise} \end{cases}$$

Ideal impulse response by linearity and antitransform:

$$h_k = \mathcal{F}^{-1}(H) = \mathcal{F}^{-1}(\mathrm{Lowpass}_{[0,\theta_2]}) - \mathcal{F}^{-1}(\mathrm{Lowpass}_{[0,\theta_1]})$$

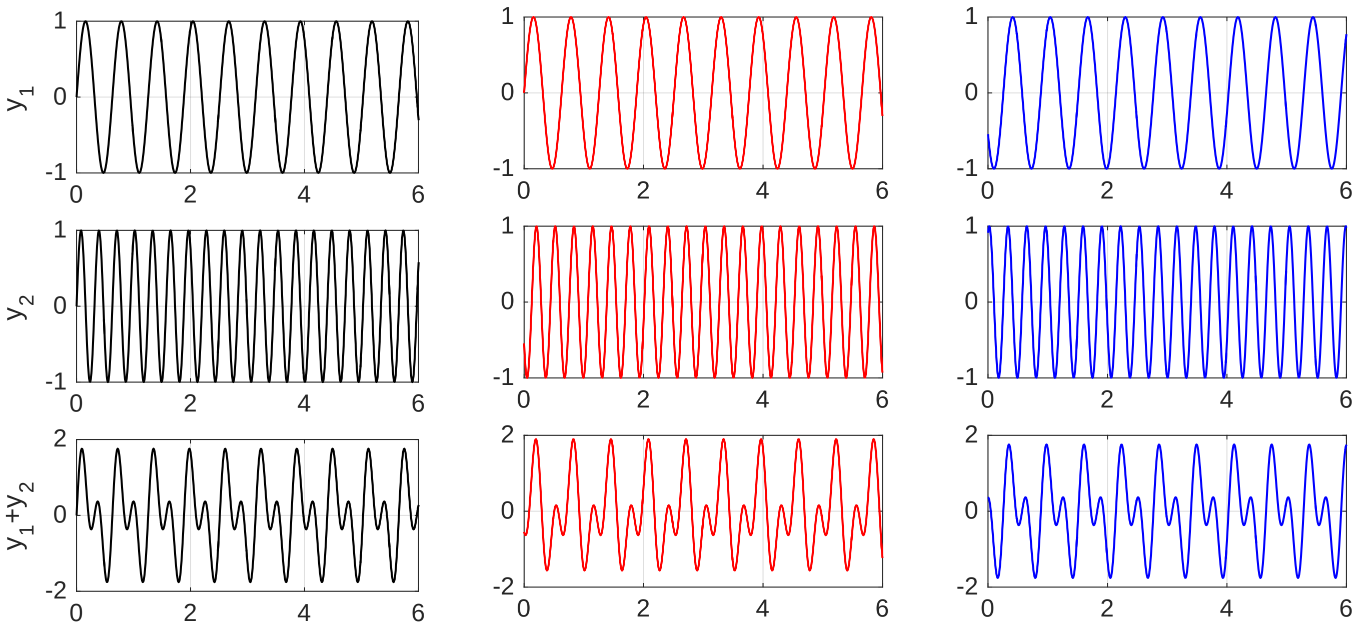
$$= \frac{\sin(k\theta_2)}{\pi k} - \frac{\sin(k\theta_1)}{\pi k}$$

Final filter by delay and multiplication with desired window w_k:

$$g_k = h_{k-N/2} w_k$$

Effects of FIR linear vs. phase

$$G(e^{j\theta}) = |G(e^{j\theta})| e^{-j\theta \frac{N}{2}}$$



No filter | Nonlinear phase filter | Linear phase filter

Achieving linear phase

Linear phase $G(e^{j\theta}) = |G(e^{j\theta})|e^{-j\theta\frac{N}{2}}$ is achieved if $\mathbf{g_k} = \mathbf{g_{N-k}}$.

$$\begin{aligned}
G(e^{j\theta}) &= \sum_{k=0}^{N} g_k e^{-j\theta k} \\
&= g_0 e^{-j\theta 0} + g_N e^{-j\theta N} + g_1 e^{-j\theta 1} + g_{N-1} e^{-j\theta (N-1)} + \dots \\
&= e^{-j\theta\frac{N}{2}} \Big(\underbrace{g_0 e^{j\theta\frac{N}{2}} + g_N e^{-j\theta\frac{N}{2}}}_{2g_0 \cos(\theta\frac{N}{2}) \text{ is real}} + \underbrace{g_1 e^{j\theta(\frac{N}{2}-1)} + g_{N-1} e^{-j\theta(\frac{N}{2}-1)}}_{2g_1 \cos(\theta(\frac{N}{2}-1)) \text{ is real}} + \dots \Big)
\end{aligned}$$

Note: *window method gives linear phase filters*. The coefficients w_k of all window functions satisfies $w_k = w_{N-k}$. If the desired impulse response h_k is also symmetric $h_k = h_{N-k}$ then $g_k = h_k w_k$ is symmetric, that is, the resulting filter has linear phase.

Course outline

- we completed the digital filtering part of the course
- next 3 lectures: Discrete Fourier Transform
- you can now do Questions 1-9 in Examples Paper 2