# 3F1 Signals and Systems: Handout 8

## Algebraic continuous to discrete mappings

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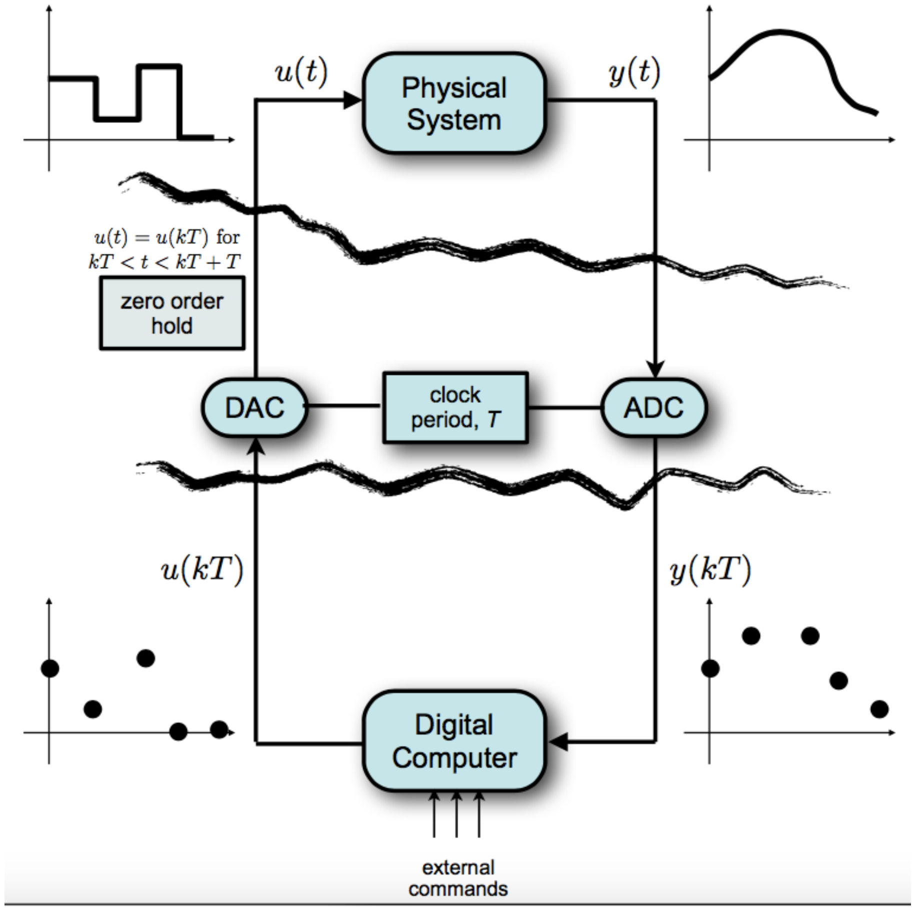
## Outline

In this lecture we will learn how to map between the (continuous) Laplace domain and the (discrete-time) $z$ domain.

Why? Two main reasons. . .

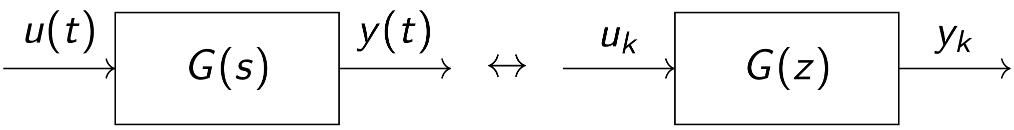
1. **converting analog designs to discrete-time:** if we have ready-made designs in the Laplace (continuous) domain, for example a library of good filters, we would like to transfer those designs to the (discrete time) $z$ domain
2. **hybrid analog/digital systems:** for systems with digital and analog components, for example a digital controller for an analog mechanical or electrical device, we need to be able to model the analog component (included in the circuit via an D/A and an A/D converter) as a discrete-time component in order to work out the overall discrete-time system stability and behaviour

# Hybrid Analog/Digital systems



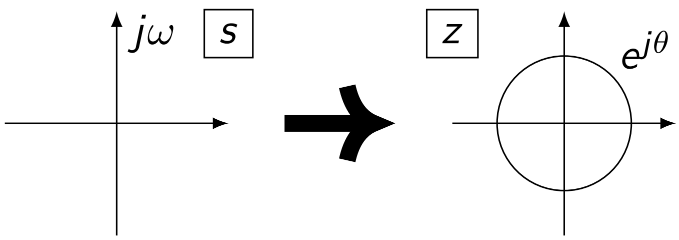



# Mapping analog systems to discrete time systems




We can map analog designs to discrete time in two ways:

1. **algebraic transformations:** transform the Laplace $s$ domain directly to the $z$ domain (this lecture)




2. **response matching:** design a discrete time system whose response matches the impulse/step/ramp response of the analog system (next lecture)

## Algebraic transformations: motivation and linear approximation

Recall that the mapping between $s$ and $z$ planes is

$$z \longrightarrow e^{sT}$$

For small $T$, can approximate this as

$$e^{sT} = 1 + sT + \frac{(sT)^2}{2} + o(T^3)$$

- a linear approximation yields $z = 1 + sT$ or

$$s = \frac{z-1}{T}$$

We can replace every occurrence of $s$ in a continuous time system by $(z-1)/T$, e.g.,

$$\frac{s-1}{s+1} \longrightarrow \frac{\frac{z-1}{T} - 1}{\frac{z-1}{T} + 1} = \frac{z-1-T}{z-1+T}$$



## Algebraic transformations: quadratic approximation

A quadratic approximation would be desirable but...

- how do we invert $z = 1 + sT + \frac{(sT)^2}{2}$ to obtain an expression for $s$?
- replacing every $s$ by a quadratic expression in $z$ would lead to an explosion in the number of poles/zeros. Simple designs in the $s$ domain would lead to higher order designs in the $z$ domain

# Common algebraic transformations

$$H(z) = H_c(s)_{s=\psi(z)} \text{ where } \psi(\cdot) \text{ is given by}$$

1. **Euler's method or Forward difference**
$$s = \frac{z-1}{T} \qquad \text{(intuition: linear approximation)}$$

2. **Backward difference**
$$s = \frac{1-z^{-1}}{T} \qquad (\text{intuition } \dot{x} \simeq \frac{x(t)-x(t-T)}{T})$$

3. **Bilinear (Tustin's) transformation**
$$s = \frac{2}{T}\frac{z-1}{z+1} \qquad (\text{or simply } \frac{z-1}{z+1})$$



# Tustin transform: motivation

We can invert the expression $s = \frac{2}{T}\frac{z-1}{z+1}$, $\frac{sT}{2}(z+1) = z-1$, and hence

$$z = \frac{1+\frac{sT}{2}}{1-\frac{sT}{2}}$$

For small $T$, we use the binomial theorem to write

$$\begin{aligned} z &= \left(1+\frac{sT}{2}\right)\left(1+\frac{sT}{2}+\frac{(sT)^2}{4}+o(T^3)\right) \\ &= 1+sT+\frac{(sT)^2}{2}+o(T^3) \end{aligned}$$

This coincides with the quadratic expansion of $e^{sT}$

# Tustin transform: motivation (continued)

Example:

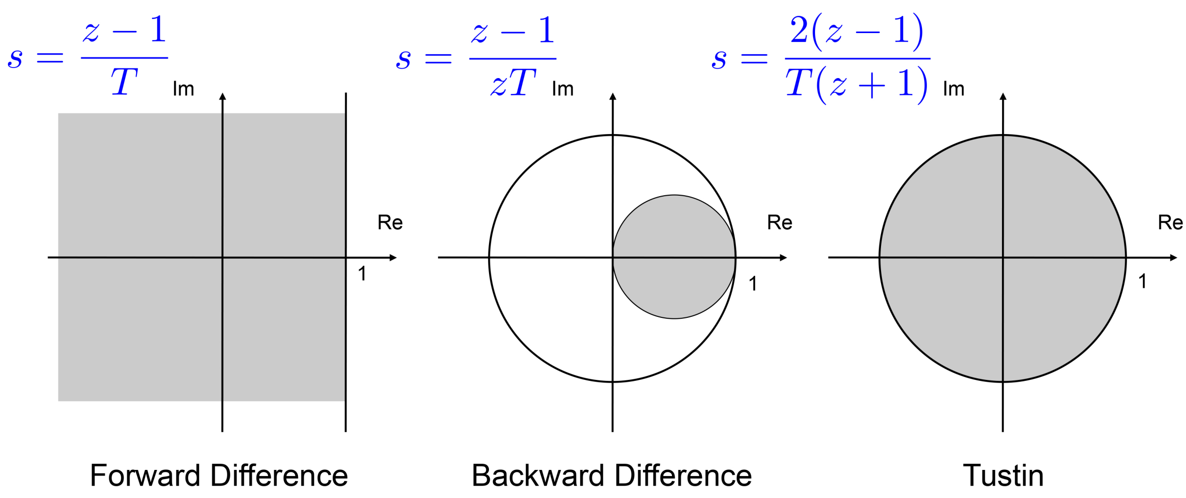
$$\frac{s-1}{s+1} \longrightarrow \frac{\frac{2}{T}\frac{z-1}{z+1}-1}{\frac{2}{T}\frac{z-1}{z+1}+1} = \frac{\left(\frac{2}{T}-1\right)z-\left(\frac{2}{T}+1\right)}{\left(\frac{2}{T}+1\right)z-\left(\frac{2}{T}-1\right)}$$

- same number of poles/zeros in the $z$ domain as in the $s$ domain!

The Tustin transform is a "trick" to get a quadratic approximation without inverting a quadratic and without leading to an increase in the number of poles/zeros when going from continuous to discrete



Each of these transformations corresponds to a certain mapping between $s-$plane and $z-$plane. Below, the shaded region shows the set of points in the $z-$plane which corresponds to the left half of the $s$-plane (stable regions)




Backward difference and Tustin transformations applied to stable continuous systems result in stable discrete time systems (all the left plane poles get mapped into the unit disk). Not necessarily true for Euler's method.

Example: **first order low pass filter** $G_c(s) = \dfrac{1}{s+1}$. Sampling $T$.

Forward (possibly unstable)

$$G(z) = G_c\left(\frac{z-1}{T}\right) = \frac{T}{z+(T-1)} \qquad \text{pole } |T-1| > 1?$$

Backward (stable)

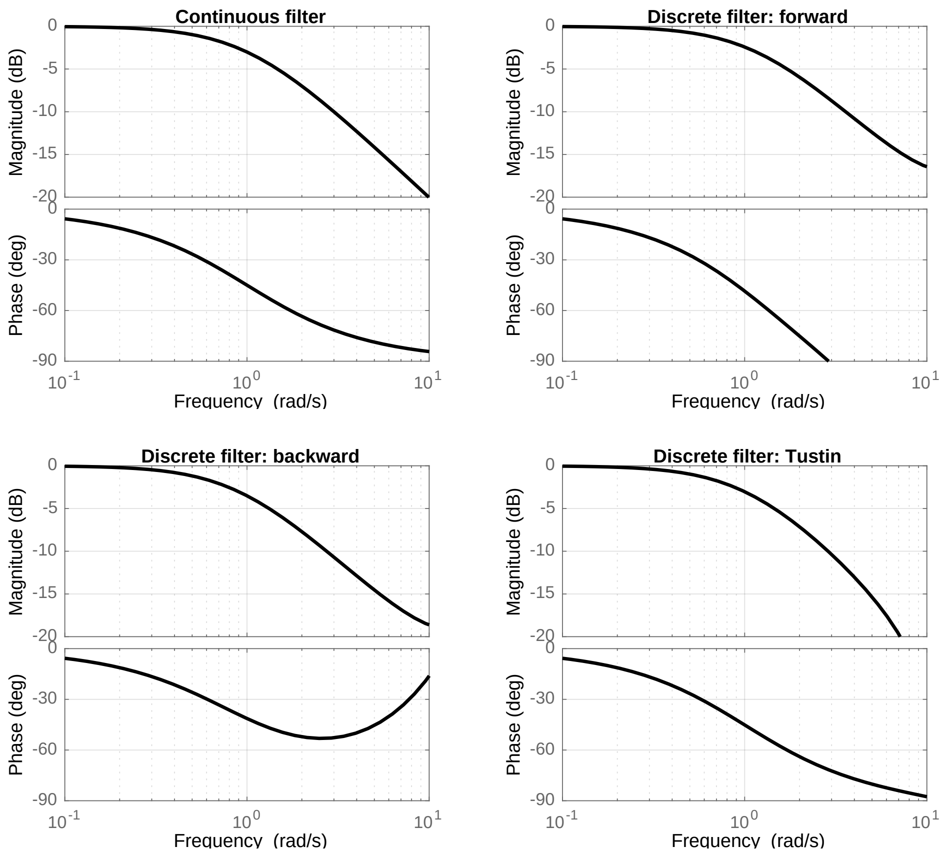
$$G(z) = G_c\left(\frac{z-1}{zT}\right) = \frac{\frac{T}{1+T}z}{z-\frac{1}{(1+T)}} \qquad \text{pole } \left|\frac{1}{1+T}\right| < 1$$

Tustin (stable)

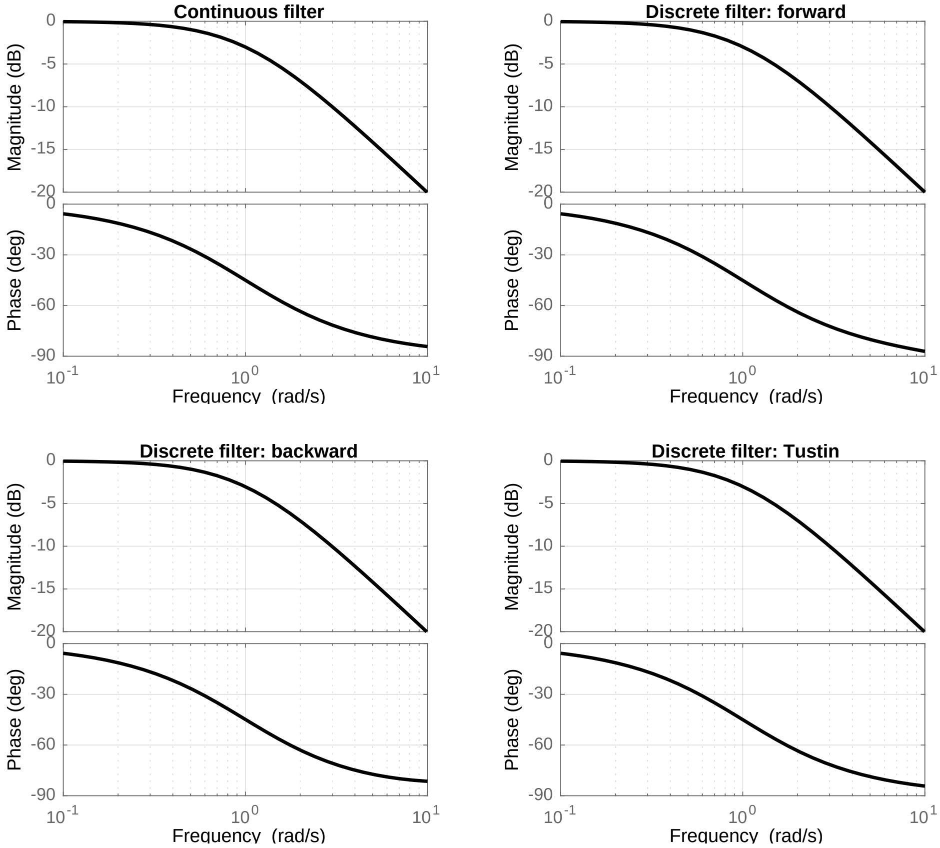
$$G(z) = G_c\left(\frac{2}{T}\frac{z-1}{z+1}\right) = \frac{\frac{T}{T+2}(z+1)}{z+\frac{T-2}{T+2}} \qquad \text{pole } \left|\frac{T-2}{T+2}\right| < 1$$



Frequency response distortion of discretized filters.
Sampling $T = 0.25$ ($\omega_{max} = \frac{\pi}{T} = 4\pi$ rad/s).

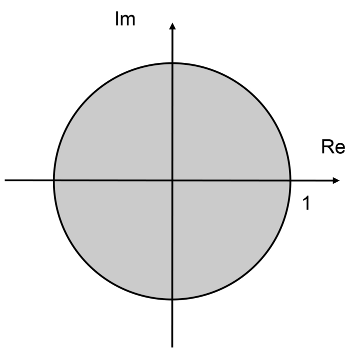


The frequency response is recovered as $T \to 0$.
Sampling $T = 0.01$ ($\omega_{max} = \frac{\pi}{T} = 100\pi$ rad/s).






# Bilinear transform: stability is preserved




$$s = \psi(z) = \frac{2}{T}\frac{z-1}{z+1}$$

Solve for $z$: $z = \psi^{-1}(s) = \dfrac{1 + sT/2}{1 - sT/2}$

For $sT/2 = \lambda + j\omega$:

$$|z|^2 = zz^* = \psi^{-1}(s)(\psi^{-1}(s))^* = \frac{(1+\lambda)^2 + \omega^2}{(1-\lambda)^2 + \omega^2}$$

If $\lambda = 0$ then

$$|z|^2 = \frac{1+\omega^2}{1+\omega^2} = 1 \;\to\; \text{unit circle}$$

If $\lambda < 0$ then $(1+\lambda)^2 < (1-\lambda)^2$ thus

$$|z|^2 = \frac{(1+\lambda)^2 + \omega^2}{(1-\lambda)^2 + \omega^2} < 1 \;\to\; \text{inside unit circle}$$

# Bilinear transform: frequency warping

$$s = \psi(z) = \frac{2}{T}\frac{z-1}{z+1}$$

Analog prototype filter: $G_c(s)$. Frequency response $G_c(j\omega)$.
Digital filter: $G(z) = G_c(\psi(z))$.

The normalized frequency response of the digital filter ($|\theta| \leq \pi$) is given by

$$G(e^{j\theta}) = G_c(\psi(e^{j\theta}))$$

where

$$\psi(e^{j\theta}) = \frac{2}{T}\frac{e^{j\theta}-1}{e^{j\theta}+1} = \frac{2}{T}\frac{e^{j\theta/2}-e^{-j\theta/2}}{e^{j\theta/2}+e^{-j\theta/2}} = \frac{2}{T}\frac{j\sin(\theta/2)}{\cos\theta/2} = j\frac{2}{T}\tan(\theta/2)$$

that is

$$G(e^{j\theta}) = G_c\left(j\frac{2}{T}\tan(\theta/2)\right) \qquad \text{frequency warping}$$



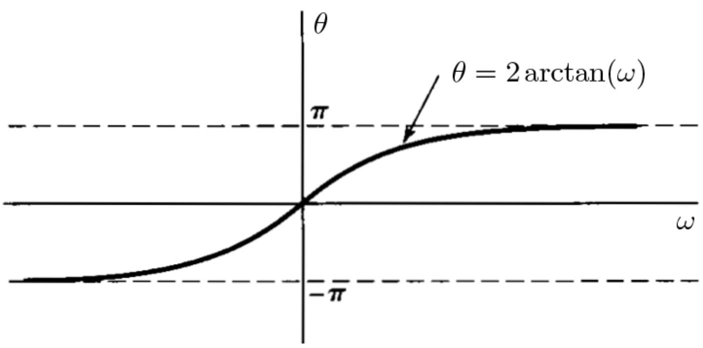
# Bilinear transform: frequency warping (continued)

$$G(e^{j\theta}) = G_c\left(j\frac{2}{T}\tan(\theta/2)\right)$$

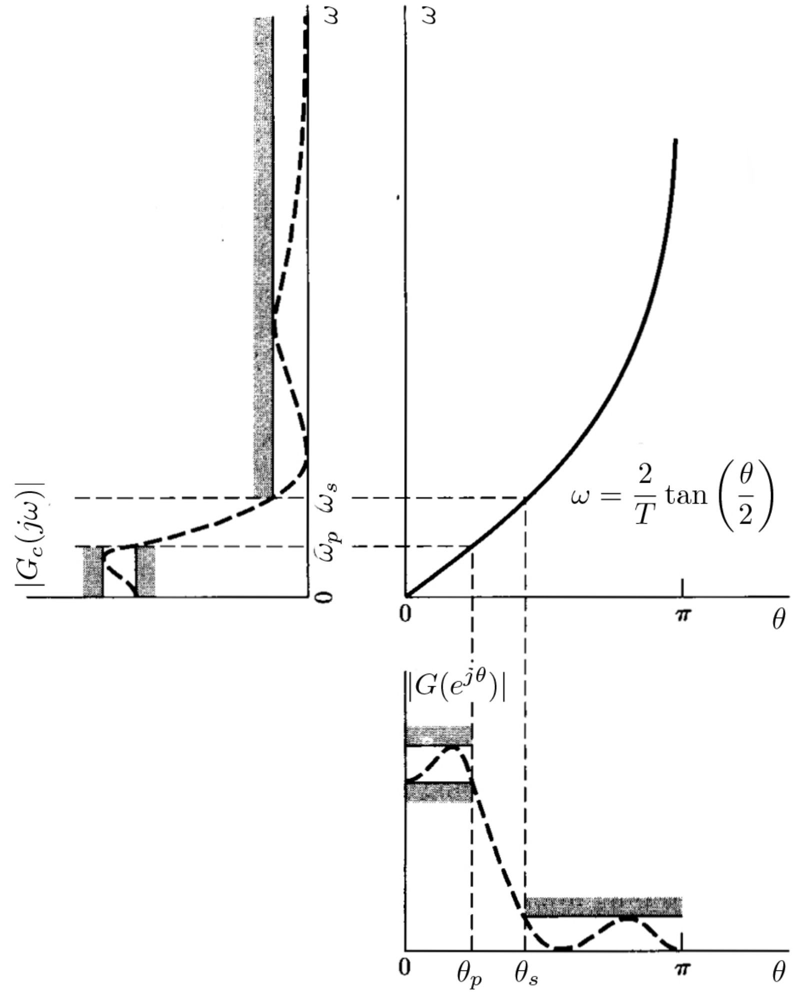
Inverse relation: the frequency response of the analog filter at $\omega$ is mapped into the frequency response of the digital filter at $\theta = 2\arctan(\omega T/2)$.

$$\omega = \frac{2}{T}\tan(\theta/2) \rightarrow \theta = 2\arctan(\omega T/2)$$

$$G_c(j\omega) = G\left(e^{j2\arctan(\omega T/2)}\right)$$



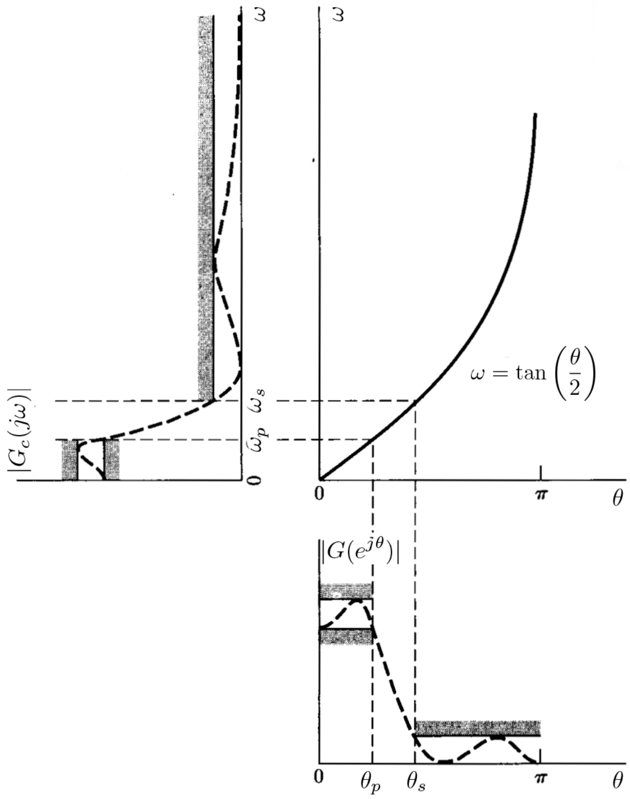
$$s = \frac{2}{T}\frac{z-1}{z+1}$$






Sometimes the $2/T$ normalisation is omitted...

$$s = \frac{z-1}{z+1}$$




It works just as well but the resulting pre-warped frequencies are "weird" (very different from the desired frequencies)

## Example (low pass filter design)

Design a first order lowpass digital filter with $-3$dB frequency of 1kHz and a sampling frequency of 8kHz.

Consider the first order analogue lowpass filter

$$G_c(s) = \frac{1}{1 + \frac{s}{\omega_c}}$$

which has gain 1 (0dB) at $s = j0$ and gain 0.5 ($-3$dB) at $s = j\omega_c$ rad/s (cutoff frequency). Thus, the normalized digital cutoff frequency reads

$$\theta_c = 2\pi \frac{1000}{8000} = \pi/4$$

The equivalent pre-warped analogue filter cutoff frequency:
With normalisation:

$$\omega_c = \frac{2}{T}\tan(\theta_c/2) = 16000\tan(\pi/8) = 6627.4 \text{ rad}\cdot s^{-1} \quad f_c = 1054.8 \text{ Hz}$$

Without normalisation:

$$\omega_c = \tan(\theta_c/2) = \tan(\pi/8) = 0.4142 \text{ rad}\cdot s^{-1} \quad f_c = 0.066 \text{ Hz}$$



## Example (low pass filter design)

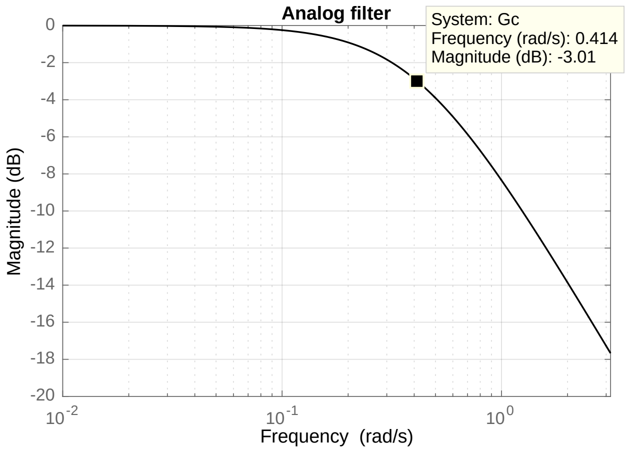
Apply now the bilinear transform $s = \psi(z) = \frac{z-1}{z+1}$.

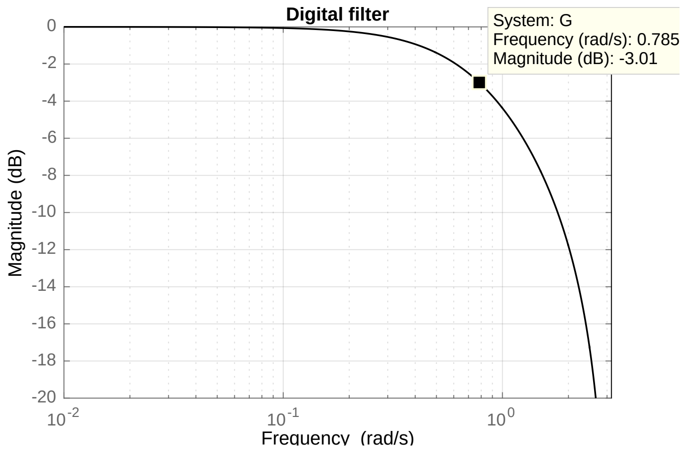
$$\begin{aligned} G(z) &= G_c(\psi(z)) = \frac{1}{1 + \frac{\psi(z)}{\omega_c}} = \frac{1}{1 + \frac{z-1}{(z+1)\omega_c}} \\ &= \frac{(z+1)\omega_c}{(z+1)\omega_c + z - 1} = \frac{(z+1)\omega_c}{(1+\omega_c)z + (\omega_c - 1)} \\ &= \frac{(z+1)\frac{\omega_c}{(1+\omega_c)}}{z + \frac{(\omega_c-1)}{(1+\omega_c)}} = \frac{0.2929(z+1)}{z - 0.4142} \end{aligned}$$

whose implementation reads

$$y_k = 0.4142 y_{k-1} + 0.2929(u_k + u_{k-1})$$

# Example (low pass filter design)






# Outline

- You can now attempt questions 1, 5, 6, 7 in Examples Paper 2
- Next lecture we will discuss response matching (another way of interfacing between analogue and discre-time)
- We will then move on to filter design