3F1 Signals and Systems: Handout 7

Nyquist stability criterion (part II: examples)

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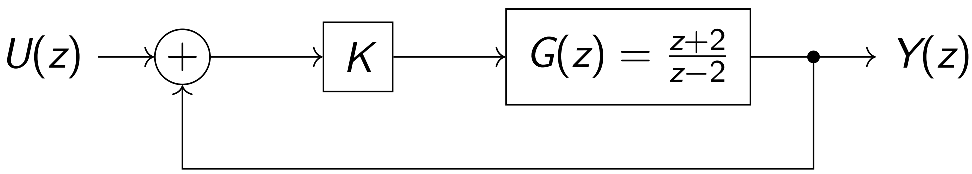
Example closed loop system



For what values of K is this closed loop stable?

Closed loop transfer function:

$$H(z) = \frac{Y(z)}{U(z)} = \frac{KG(z)}{1 + KG(z)}$$

Nyquist stability crtierion:

The closed loop system is stable if and only if the number of encirclements of $-1/K$ by $G(e^{j\theta})$ as θ increases from $-\pi$ to π equals the number of open loop poles $N_{op,o}$ strictly outside the unit circle.

Methodology

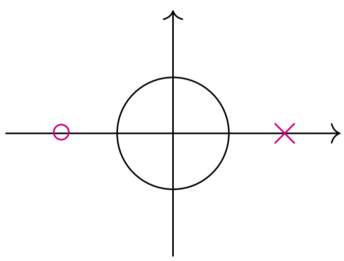
1. draw a pole-zero diagram of the open loop system $G(z)$
2. count poles strictly outside the unit circle: this is the number $\mathcal{C}$ of encirclements of $-1/K$ necessary to achieve stability
3. draw the Nyquist locus of $G(e^{j\theta})$ for θ from 0 to π (this is always given to you in examples and exams)
4. work out the values of $G(e^{j\theta})$ for $\theta = 0$, $\theta = \pi$ and any key points of interest and annotate those on the graph. You may want to annotate the "direction" of the curve from 0 to π.
5. **complete** the graph for θ from π to 2π (or, equivalently, $-\pi$ to 0) by symmetry (mirror around the real axis, assuming an open loop system with a real delta response.)
6. if the open loop system has poles on the unit circle, you can work out the asymptotes as the locus goes to infinity, and how the curves connect "at infinity" (more about this later)
7. apply the Nyquist stability criterion to determine range(s) of K for which the closed loop system is stable

Example

1. draw a pole-zero diagram of the open loop system $G(z)$
2. count poles strictly outside the unit circle: this is the number $\mathcal{C}$ of encirclements of $-1/K$ necessary to achieve stability

$$G(z) = \frac{z+2}{z-2}$$

Pole-zero diagram:



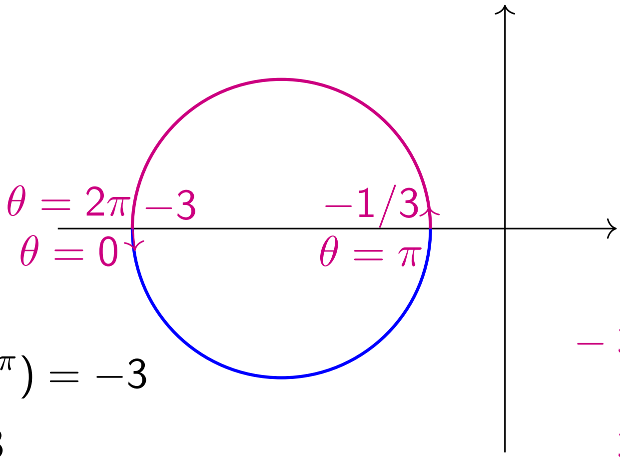
$$G(e^{j0}) = G(1) = -3$$
$$G(e^{j\pi}) = G(-1) = -1/3$$

One pole outside the unit circle $\longrightarrow$ $\mathcal{C} = 1$

Example

4. work out the values of $G(e^{j\theta})$ for $\theta = 0$, $\theta = \pi$ and any key points of interest and annotate those on the graph. Annotate the "direction" of the curve from 0 to π.
5. **complete** the graph for θ from π to 2π by symmetry

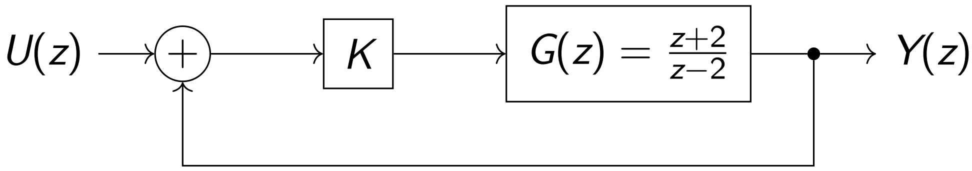
Nyquist diagram:



$$G(e^{j0}) = G(e^{j2\pi}) = -3$$
$$G(e^{j\pi}) = -1/3$$
$$\mathcal{C} = 1$$

$$-3 < -\frac{1}{K} < -\frac{1}{3}$$
$$3 > \frac{1}{K} > \frac{1}{3}$$
$$\mathbf{\frac{1}{3} < K < 3}$$

Example (continued)



We've established that the system is stable for $\mathbf{\frac{1}{3} < K < 3}$.

Let's check for example $\mathbf{K = 1}$. The closed-loop transfer function is

$$H(z) = \frac{KG(z)}{1 + KG(z)} = \frac{\frac{z+2}{z-2}}{1 + \frac{z+2}{z-2}} = \frac{z+2}{z-2+z+2} = \frac{1}{2} + z^{-1}$$

This is an FIR with delta response $(1/2, 1)$ and evidently **stable**.

Let's check for example $\mathbf{K = 4}$. The closed-loop transfer function is

$$H(z) = \frac{KG(z)}{1 + KG(z)} = \frac{4(z+2)}{z-2+4(z+2)} = \frac{4(z+2)}{5z+6} = \frac{4}{5} \cdot \frac{z+2}{z+\frac{6}{5}}$$

This has a pole at $z = -6/5$ outside the unit circle, so **unstable**

Proof of the Nyquist stability criterion: canonical form

- in the proof, we restricted our attention to canonical forms

$$G(z) = \frac{b_0 + b_1 z^{-1} + \ldots b_m z^{-m}}{1 + a_1 z^{-1} + \ldots + a_n z^{-m}}$$

  giving a fraction of polynomials of the same degree $\max(m, n)$
- we excluded $b_0 = -1/K$ as this would cause $1 + KG(z)$ to lose degree in its numerator (more about it on the next few slides)
- in fact, the theory also applies to systems for which $b_0 = 0$. These are "predictor" systems for which the output y_k depends on past inputs $y_{k-1}, y_{k-2}, \ldots$ and past inputs $x_{k-1}, x_{k-2}, \ldots$ but *not* on the current input x_k
- for these systems, $n = \deg\{a(z)\} > m = \deg\{b(z)\}$, but the numerator and $a(z) + Kb(z)$ and denominator $a(z)$ of $1 + KG(z)$ still have the same degree n.

Proof of the Nyquist stability criterion: $K = -1/b_0$ case

- What happens if $K = -1/b_0$?
- Remember that, for an open loop transfer function $G(z) = \frac{b(z)}{a(z)}$, we get a closed loop transfer function

$$H(z) = \frac{KG(z)}{1 + KG(z)} = \frac{Kb(z)}{a(z) + Kb(z)}$$

- if $K = -1/b_0$, the denominator degree n is smaller than the numerator degree m! This an **illegal** transfer function because it doesn't correspond to the delta response of a system via the one-sided z transform (polynomial division gives terms in positive powers of z...)
- Is this a problem? Is there a risk that the value $K = -1/b_0$ is within the range that we declare to be stable?

Proof of the Nyquist stability criterion: $K = -1/b_0$ case

- Let's assume a K close to but not equal to $-1/b_0$, i.e., $K + 1/b_0 = \varepsilon$ where ε is very small
- The denominator will be a polynomial of the form

$$\begin{aligned} a(z) + Kb(z) &= (1 + Kb_0)z^n + (a_1 + Kb_1)z^{n-1} + \dots \\ &= \varepsilon b_0 z^n + (a_1 + Kb_1)z^{n-1} + \dots \end{aligned}$$

hence

$$H(z) = \frac{Kb(z)}{a(z) + Kb(z)} = \frac{1}{\varepsilon b_0} \cdot \frac{Kb(z)}{z^n + \frac{1}{\varepsilon b_0}((a_1 + Kb_1)z^{n-1} + \dots)}$$

- All except the leading coefficient of the denominator polynomial, and in particular its constant term scale with $\frac{1}{\varepsilon}$
- The only way this can happen is if at least one pole is going to infinity.
- Hence, $K = -1/b_0$ can only be in the unstable region of the Nyquist diagram since at least one pole of the closed loop transfer function is approaching infinity (and hence lies far outside the unit circle)

Nyquist stability criterion: strictly inside/outside?

- for the proof of Nyquist, we included zeros/poles on the unit circle as being "inside" the unit circle
- that's because zeros/poles on the unit circle cause "encirclements" (via infinity) same as zeros/poles inside the unit circle
- however in the last step of the proof, when it comes to expressing the stability of the closed loop system in terms of encirclements, it is not 100% correct to exlude only closed loop poles that are strictly outside the unit circle. Poles on the unit circle also correspond to an unstable open loop pole
- Luckily, this is easy to exclude: closed loop poles will be on the unit circle exactly on the boundary of the stable ranges of K
- **the inequalities defining the stable range of K are...**

strict inequalities

Discrete vs. continuous Nyquist diagram

- if the open loop has no poles strictly oustide the unit circle, $\mathcal{C} = 0$ and the rule remains "leave $-1/K$ on your left as you move along the Nyquist diagram"
- although plotting the discrete time Nyquist diagram is quite different from its continuous time equivalent, its interpretation is similar: gain and phase margins can be read off in the same way, and have the same meanings. Gain and phase margins can also be found from the Bode diagram in the same way as for continuous time

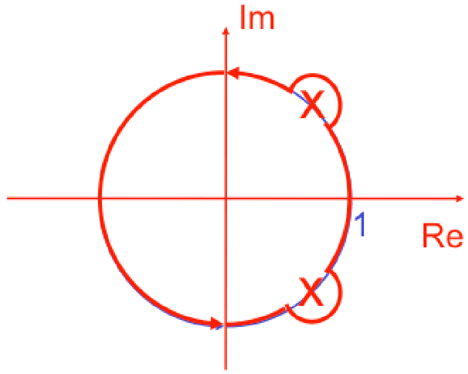
What about open loop poles **on** the unit circle?

- the open loop poles on the unit circle will cause the Nyquist diagram $G(e^{j\theta})$ to go to infinity as it passes through the pole on the unit circle
- for the closed loop,

$$H(z) = \frac{KG(z)}{1 + KG(z)}$$

  tends to 1 around the open loop pole
- Hence, the open loop pole should be counted as being "inside" the unit circle from a Nyquist criterion point of view.
- How do we draw the Nyquist diagram going to infinity, and how do we account for those poles as if they were inside the unit circle?

Open-loop poles on the unit circle



Closing the locus

In order to obtain a closed curve for the Nyquist locus and hence correctly count the encirclements, it is customary to indent the path of z around the poles on the unit circle with a small semi-circular excursion outside the unit circle. Then the open loop poles on the unit circle are counted as being stable in the stability criterion.

The "right turn" in the z-plane then gives a "right turn" in the G-plane and a circular arc of large radius is produced which continues for $m\pi$ radians where m is the multiplicity of the pole on the unit circle.

Asymptotes

If there is an open-loop pole of multiplicity one on the unit circle then the Nyquist diagram will be asymptotic to a straight line as it tends to infinity. It is possible to find the asymptote along which it tends as follows.

Suppose that $G(z)$ has a pole at $z = 1$, i.e.

$$G(z) = \frac{1}{(z-1)}\ F(z)$$

where $F(z)$ has no poles or zeros at $z = 1$.

Then for $z \approx 1$, expand $F(z)$ in a Taylor series to give

$$\begin{aligned} G(z) \quad &= \quad \frac{1}{(z-1)}\left\{F(1) + F'(1)(z-1) + \right. \\ &\qquad \left. + \frac{1}{2}F''(1)(z-1)^2 + \dots \right\} \end{aligned}$$

$$\approx \frac{F(1)}{(z-1)} + F'(1)$$

But

$$\begin{aligned}
\frac{1}{e^{j\theta}-1} &= \left\{e^{j\theta/2}(e^{j\theta/2} - e^{-j\theta/2})\right\}^{-1} \\
&= \frac{e^{-j\theta/2}}{2j\sin(\theta/2)} \\
&= -\frac{1}{2} - \frac{j}{2\tan(\theta/2)}
\end{aligned}$$

$$e^{-j\theta/2} = \cos\frac{\theta}{2} - j\sin\frac{\theta}{2}$$

Hence

$$G(e^{j\theta}) \approx -\frac{1}{2}F(1) + F'(1) - j\overbrace{\frac{F(1)}{2\tan(\theta/2)}}^{\text{large as } \theta \to 0}$$

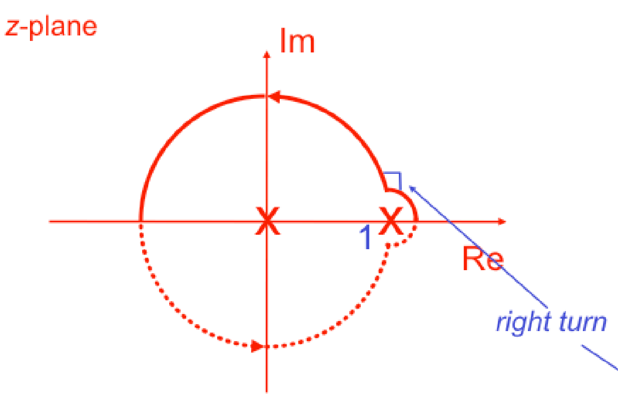
Hence the asymptote as $\theta \to 0$ will be a straight line with a constant real part of $-\frac{1}{2}F(1) + F'(1)$.

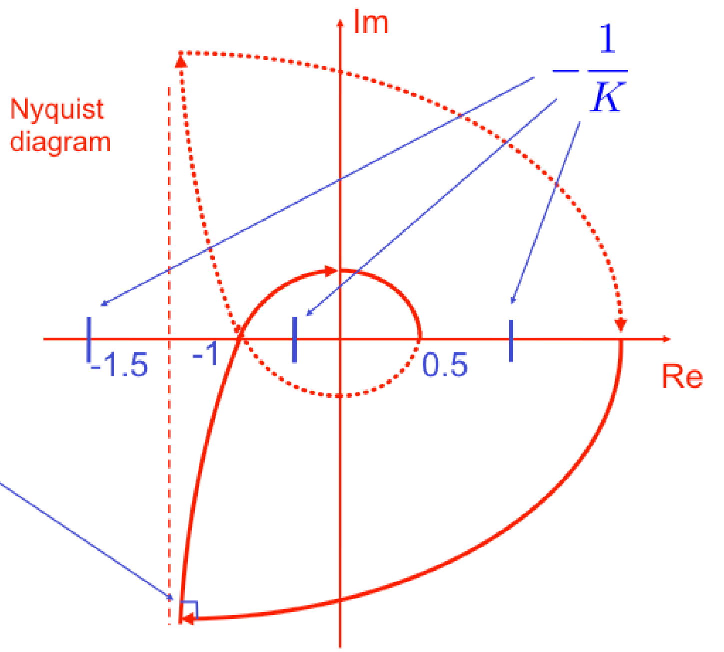
For multiple poles on the unit circle the asymptotic behaviour is more complex and requires more terms in the Taylor series expansion.

Example.

$$G(z) = \frac{1}{z(z-1)}. \quad \text{Then, } F(z) = \frac{1}{z} \;, \; F'(z) = -\frac{1}{z^2}$$

$$\text{and } -\frac{1}{2}F(1) + F'(1) = -1.5$$





Closed-loop stable for

$$-\frac{1}{K} < -1 \quad \Leftrightarrow 0 < K < 1$$