

3F1 Signals and Systems: Handout 6

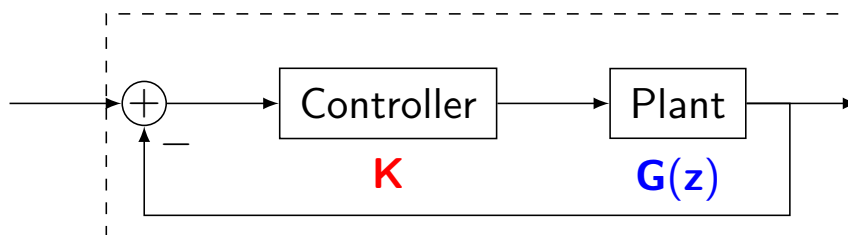
Nyquist stability criterion (part I: theory)

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1 / 22

Closed loop control systems



Given K and $G(z)$, is the closed loop stable?

- ▶ we don't assume that the open loop system is stable
- ▶ feedback can be used to stabilise an unstable system
- ▶ feedback can also be used to improve the stability and convergence of a stable system
- ▶ we do assume that the open loop system is **causal** in this lecture

2 / 22

What is a plant?

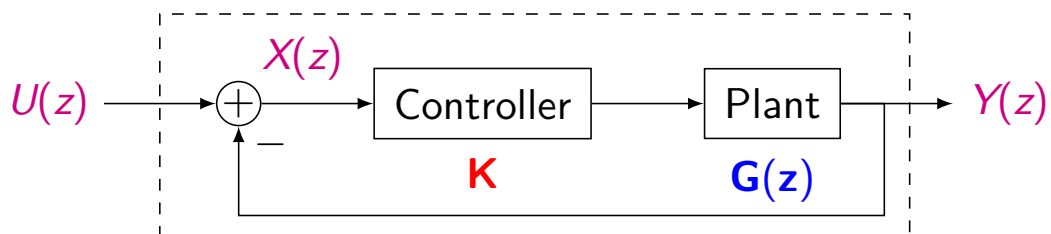


Photos created using ChatGPT

- ▶ I don't know what control theorists mean by "plant" ...
- ▶ I think the word "plant" is a control theory word for "a thing that needs to be stabilised" ...
- ▶ I assume the word has its meaning in "industrial plant" (photo on the right) rather than "house plant" (photo on the left)
- ▶ We will only be interested in stabilising linear systems in this course. For us, a "plant" is a linear system!

3 / 22

Closed loop control systems



$$X(z) = U(z) - Y(z) = U(z) - KG(z)X(z)$$

$$U(z) = (1 + KG(z))X(z)$$

$$X(z) = \frac{U(z)}{1 + KG(z)}$$

$$Y(z) = KG(z)X(z) = \frac{KG(z)}{1 + KG(z)} U(z)$$

and hence the transfer function of the closed loop system is

$$H(z) = \frac{Y(z)}{U(z)} = \frac{KG(z)}{1 + KG(z)}$$

4 / 22

The Nyquist diagram

The “Nyquist stability criterion” predicts the stability of the closed loop

$$H(z) = \frac{KG(z)}{1 + KG(z)}$$

from the “Nyquist diagram” of the open loop, i.e.,

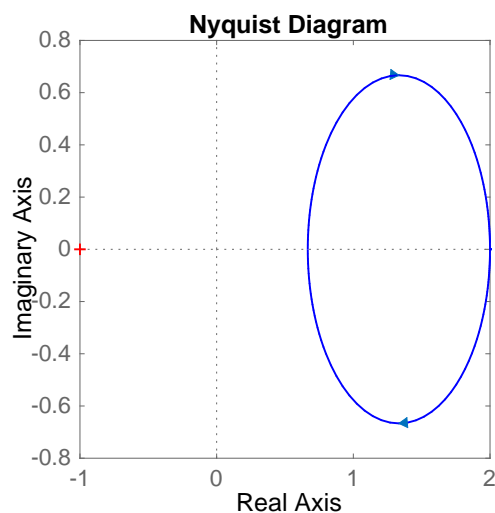
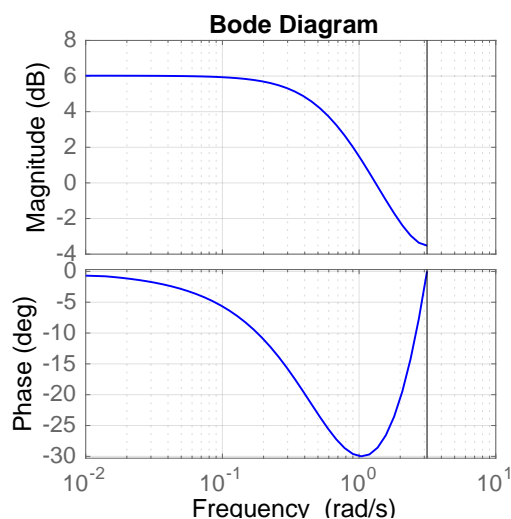
$G(e^{j\theta})$ drawn in the complex plane (Argand diagram) as θ increases from 0 to 2π (or, alternatively, from $-\pi$ to π).

5 / 22

Drawing the Nyquist diagram for a known transfer function

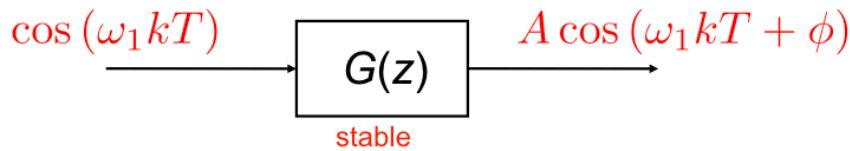
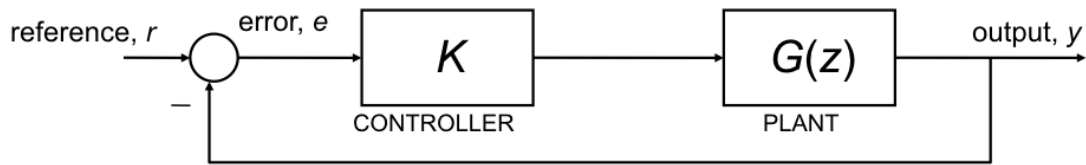
Example

$$G(z) = \frac{1}{1 - 0.5z^{-1}} = \frac{z}{z - 0.5}$$

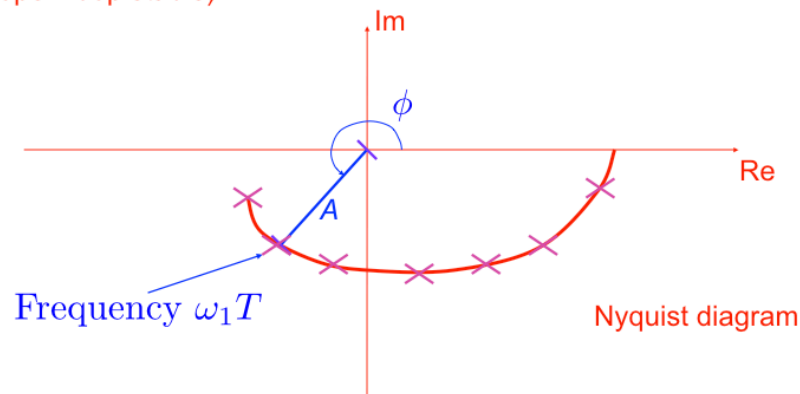


6 / 22

Drawing the Nyquist diagram in an experiment



Experiment (if open loop stable):



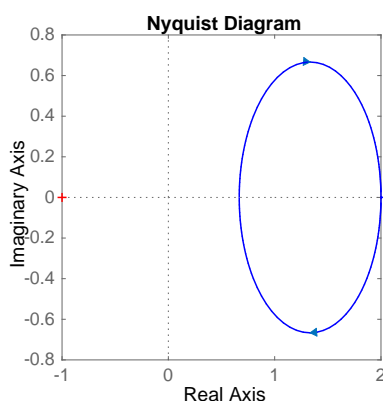
7 / 22

The Nyquist stability criterion (preview)

The closed loop system is stable if and only if the number of encirclements of $-1/K$ by $G(e^{j\theta})$ as θ increases from 0 to 2π equals the number of open loop poles strictly outside the unit circle.

Example:

$$G(z) = \frac{z}{z - 1/2}$$



- ▶ no open loop poles outside unit circle
- ▶ no encirclements allowed
- ▶ $-1/K < 2/3$ or $-1/K > 2$
- ▶ $1/K > -2/3$, or $1/K < -2$
- ▶ $K < -3/2$ or $K > -1/2$

8 / 22

Proof of the Nyquist stability criterion: outline

Over the next few pages, we will prove the Nyquist stability criterion. The individual steps to get there are simple, but it's a lengthy sequence of steps...To help you manage this, here is an outline:

1. poles/zeros and encirclements of the origin
2. counting encirclements of the origin
3. poles and zeros of $1 + KG(z)$ and corresponding encirclements
4. polynomial degrees vs. numbers of poles/zeros
5. pole arithmetic: how do closed and open loop stable/unstable poles add up?
6. shifting the graph to $-1/K$
7. The Nyquist criterion

9 / 22

The encirclement property

For any rational function $F(z)$, the number of **encirclements of the origin** by a Nyquist trajectory $F(e^{j\theta})$ as θ increases from 0 to 2π (or, equivalently, from $-\pi$ to π) tells us something about the number of poles and zeros of $F(z)$ inside the unit circle.

Why?

Consider

$$F(z) = A \frac{(z - z_1)(z - z_2) \cdots (z - z_m)}{(z - p_1)(z - p_2) \cdots (z - p_n)}$$

and remember that

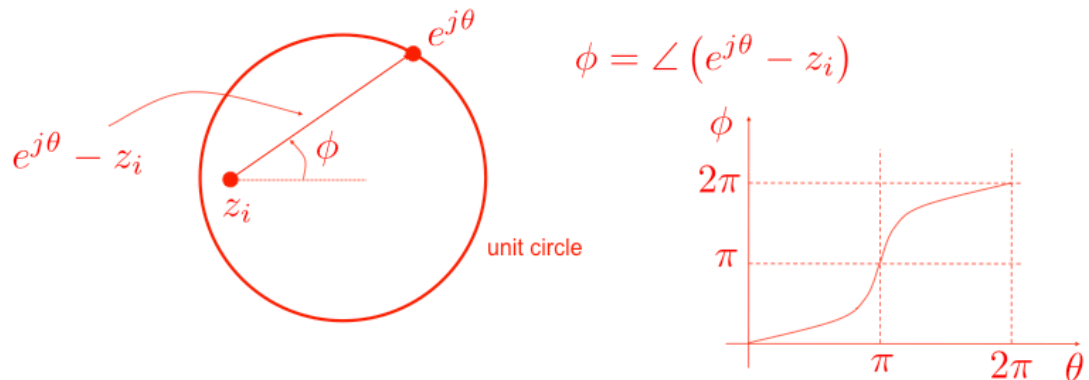
$$\angle F(e^{j\theta}) = \angle A + \sum_{i=1}^m \angle(e^{j\theta} - z_i) - \sum_{i=1}^n \angle(e^{j\theta} - p_i)$$

Note: in this lecture, when we talk of “encirclements” we **always** mean counterclockwise encirclements (in the trigonometric sense), with clockwise encirclements accounted as negative counts.

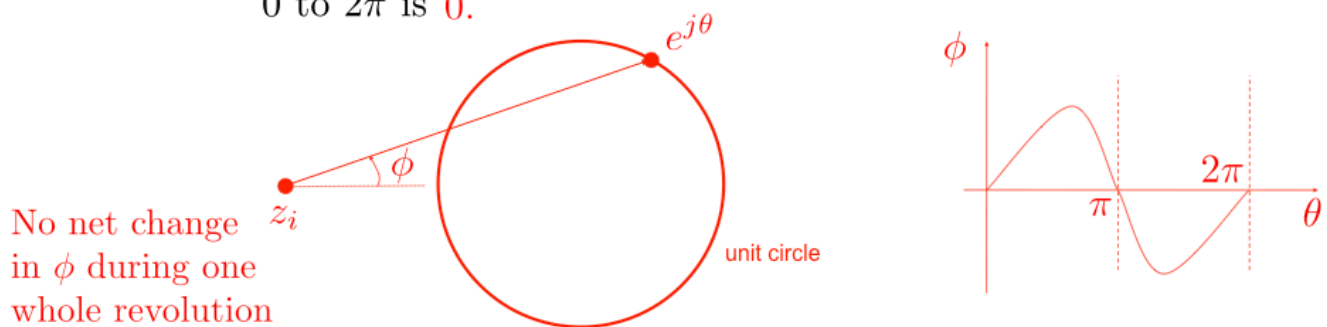
10 / 22

$$F(z) = \frac{A(z - z_1)(z - z_2) \dots (z - z_m)}{(z - p_1)(z - p_2) \dots (z - p_n)}$$

(ii) If $|z_i| < 1$ then the increase in $\angle(e^{j\theta} - z_i)$ as θ increases from 0 to 2π is 2π .



(iii) If $|z_i| > 1$ then the increase in $\angle(e^{j\theta} - z_i)$ as θ increases from 0 to 2π is 0.



11 / 22

Encirclement property in details

- ▶ Previous slide showed the effect of zeros: angle of $G(e^{j\theta})$ goes from 0 to 2π if a zero inside the unit circle (counter-clockwise encirclement)
- ▶ The effect of a pole is similar except that the angle will go from 0 to -2π (clockwise encirclement)
- ▶ What about a zero/pole **on** the unit circle?
- ▶ The phase jumps by π ($-\pi$) when it passes the zero (pole)...
- ▶ A zero (pole) **on** the unit circle **causes an encirclement** just the same as a zero (pole) inside the unit circle
- ▶ The encirclement “passes through infinity” (it escapes to the “infinite edge” of the complex plane and returns from an “infinite edge” on the diametrically opposed side of the complex plane)

12 / 22

Counting encirclements

Let

- ▶ $N_{z,i}$ be the number of zeros of $F(z)$ inside the unit circle
 - ▶ $N_{p,i}$ be the number of poles of $F(z)$ inside the unit circle
- ("inside" includes the unit circle)

Then, the overall increase α of $\angle F(e^{j\theta})$ as θ increases from 0 to 2π (or, equivalently, from $-\pi$ to π) is

$$\alpha = 2\pi(N_{z,i} - N_{p,i})$$

Hence, the number of encirclements \mathcal{C} of the origin as θ increases from 0 to 2π (or $-\pi$ to π) satisfies

$$\mathcal{C} = N_{z,i} - N_{p,i}$$

13 / 22

Closed loop stability

Remember that the transfer function of the closed loop is

$$W(z) = \frac{KG(z)}{1 + KG(z)}$$

Hence, the closed loop poles are the solutions of

$$1 + KG(z) = 0.$$

Re-writing $G(z)$ as

$$G(z) = \frac{b(z)}{a(z)}$$

we obtain

$$1 + KG(z) = 1 + K\frac{b(z)}{a(z)} = \frac{a(z) + Kb(z)}{a(z)}$$

14 / 22

Zeros and poles of $1 + KG(z)$

We have determined that

$$1 + KG(z) = \frac{a(z) + Kb(z)}{a(z)}$$

and that the closed loop poles are solutions of $1 + KG(z) = 0$.

Hence

- ▶ the **zeros** of $1 + KG(z)$ (when $a(z) + Kb(z) = 0$) are the **closed loop poles**
- ▶ the **poles** of $1 + KG(z)$ (when $a(z) = 0$) are the **open loop poles**

15 / 22

Encirclements of $1 + KG(z)$

Let

- ▶ $N_{z,i}$ be the number of zeros of $1 + KG(z)$ inside the unit circle
- ▶ $N_{p,i}$ be the number of poles of $1 + KG(z)$ inside the unit circle

This is equivalent to

- ▶ $N_{cp,i} = N_{z,i}$ is the number of **closed loop poles** inside the unit circle
- ▶ $N_{op,i} = N_{p,i}$ is the number of **open loop poles** inside the unit circle

The number \mathcal{C} of encirclements of the origin by $1 + KG(e^{j\theta})$ as θ goes from $-\pi$ to π (or, equivalently, from 0 to 2π) is

$$\begin{aligned}\mathcal{C} &= N_{z,i} - N_{p,i} \\ &= N_{cp,i} - N_{op,i}\end{aligned}$$

16 / 22

Polynomial degrees

Recall that for a linear system in canonical form,

$$\begin{aligned} G(z) &= \frac{b(z)}{a(z)} = \frac{b_0 + b_1 z^{-1} + \dots + b_m z^{-m}}{1 + a_1 z^{-1} + \dots + a_n z^{-n}} \\ &= \frac{b_0 z^{\max(m,n)} + \dots + b_m z^{\min(n-m,0)}}{z^{\max(m,n)} + \dots + a_n z^{\min(m-n,0)}} \end{aligned}$$

Now consider

$$1 + KG(z) = \frac{a(z) + Kb(z)}{a(z)}$$

In the numerator, we add two polynomials of the same degree (K is a constant and doesn't affect polynomial degrees). Hence,

$$\deg(a(z) + Kb(z)) \leq \deg(a(z)) = \max(m, n)$$

with equality unless $1 + Kb_0 = 0$. We will assume for now that $K \neq -1/b_0$, $\implies 1 + KG(z)$ is a fraction of two polynomials of degree $\ell = \max(m, n)$.

17 / 22

Pole arithmetic

Since $\deg(a(z) + Kb(z)) = \deg(a(z)) = \ell$, both the numerator and denominator of $1 + KG(z)$ have ℓ roots. Some of them may be inside the unit circle and some outside the unit circle. If we complete our list of pole counters to

- ▶ $N_{cp,i} = N_{z,i}$ counts **closed loop poles** inside the unit circle
- ▶ $N_{op,i} = N_{p,i}$ counts **open loop poles** inside the unit circle
- ▶ $N_{cp,o} = N_{z,o}$ counts **closed loop poles outside** the unit circle
- ▶ $N_{op,o} = N_{p,o}$ counts **open loop poles outside** the unit circle

we have

$$N_{cp,i} + N_{cp,o} = N_{op,i} + N_{op,o} = \ell.$$

Hence, the number of encirclements \mathcal{C} satisfies

$$\mathcal{C} = N_{cp,i} - N_{op,i} = (\ell - N_{cp,o}) - (\ell - N_{op,o}) = N_{op,o} - N_{cp,o}.$$

18 / 22

The Nyquist stability criterion (preliminary version)

The closed loop system is stable if $N_{cp,o} = 0$, i.e., there are no closed loop poles outside the unit circle. Hence, we have proved the Nyquist stability criterion:

The closed loop system is stable if and only if the number of encirclements of the origin by $1 + KG(e^{j\theta})$ as θ increases from $-\pi$ to π equals the number of open loop poles $N_{op,o}$ strictly outside the unit circle.

19 / 22

Shifting encirclements to $-1/K$

When $1 + KG(e^{j\theta})$ encircles the **origin**,

- ▶ $KG(e^{j\theta})$ encircles the point

$$-1 = e^{j\pi}$$

- ▶ $G(e^{j\theta})$ encircles

$$-1/K$$

Hence, the number \mathcal{C} of encirclements of the point $-1/K$ by $G(e^{j\theta})$ as θ goes from $-\pi$ to π (the Nyquist diagram!!) satisfies

$$\mathcal{C} = N_{cp,i} - N_{op,i} = N_{op,o} - N_{cp,o}.$$

20 / 22

The Nyquist stability criterion (final version)

The closed loop system is stable if $N_{cp,o} = 0$, i.e., there are no closed loop poles outside the unit circle. Hence, we have proved the Nyquist stability criterion:

The closed loop system is stable if and only if the number of encirclements of $-1/K$ by $G(e^{j\theta})$ as θ increases from $-\pi$ to π equals the number of open loop poles $N_{op,o}$ strictly outside the unit circle.

21 / 22

Course outline

- ▶ we have completed the proof of the Nyquist stability criterion
- ▶ next lecture we will brush up the proof
 - ▶ what happens when $b_0 = -1/K$?
 - ▶ what happens with poles on the unit circle?
- ▶ we will go through a few examples and complete the Nyquist chapter
- ▶ after that, we will leave stability behind and study signal processing

22 / 22