3F1 Signals and Systems: Handout 5

Final value theorem and Bode diagrams

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Final Value Theorem (FVT)

Suppose that all the poles of $(z-1)Y(z)$ lie strictly inside the unit circle. Then

$$\lim_{k\to\infty} y_k = \lim_{z\to 1}(z-1)Y(z)$$

Warning: you must check the poles of $(z-1)Y(z)$ before applying the FVT.

Proof

All poles of $Y(z)$ are in the unit circle except possibly a (non-repeated) pole at $z = 1$. Assuming distinct poles, we can rewrite

$$Y(z) = \frac{\alpha_0}{1 - z^{-1}} + \sum_i \frac{\alpha_i}{1 - p_i z^{-1}} \text{ where } |p_i| < 1$$

By inverse z-transform:

$$\lim_{k\to\infty} y_k = \lim_{k\to\infty} \left(\alpha_0 + \sum_i \alpha_i p_i^k \right) = \alpha_0$$

Also,

$$\lim_{z\to 1} (z-1)Y(z) = \lim_{z\to 1} \left(z\alpha_0 + (z-1) \sum_i \frac{\alpha_i}{1 - p_i z^{-1}} \right) = \alpha_0 \ .$$

(If poles are not distinct, the right-hand side of the $\sum$ is more convoluted but limits still converge to zero as $k \to \infty$ and $z \to 1$.)

Note that this is essentially the proof of the cover-up rule of partial fractions for the root $z = 1$.

Application: limit of the step response

Consider a system with transfer function $G(z)$, a step signal input $\{u_k\}_{k\geq 0} = \{1\}_{k\geq 0}$, and output $\{y_k\}_{k\geq 0}$

$$U(z) = \frac{z}{z-1}$$

Then,

$$\begin{aligned} \lim_{k\to\infty} y_k &= \lim_{z\to 1} (z-1)Y(z) \\ &= \lim_{z\to 1} (z-1)G(z)U(z) \\ &= \lim_{z\to 1} zG(z) \\ &= G(1) \end{aligned}$$

The limit of the step response as k goes to infinity is the value of the transfer function $G(z)$ for $z = 1$.

Example

Consider the filter

$$y_{k+1} + \frac{1}{2} y_k = x_{k+1}$$

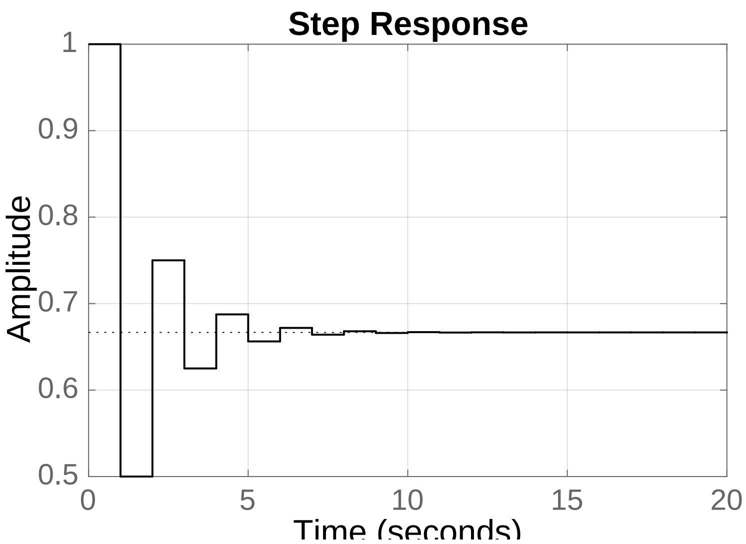
z-transform with initial $y_0 = 0$

$$\left(z + \frac{1}{2}\right) Y(z) = zX(z)$$

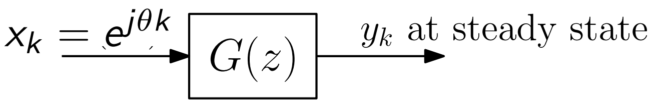
$$\Rightarrow \quad Y(z) = \underbrace{\frac{2z}{2z+1}}_{G(z)} X(z)$$

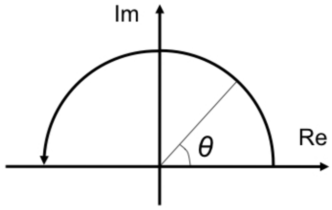
Steady-state response to a unit step input $U(z) = \frac{z}{z-1}$

$$\lim_{k \to \infty} y_k = G(1) = \frac{2}{3}$$



Steady-state response to a *complex* sinusoidal input





$$\Rightarrow X(z) = \frac{1}{1-e^{j\theta}z^{-1}} \qquad G(z) = \frac{b(z)}{(z-p_1)(z-p_2)\cdots(z-p_n)}$$

We assume a **stable** system, $|p_i| < 1$ for all i

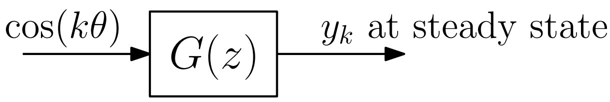
$$\Rightarrow Y(z) = G(z)X(z) = \underbrace{\frac{G(e^{j\theta})}{1-e^{j\theta}z^{-1}}}_{\text{input}} + \underbrace{\text{terms of the form } \frac{\beta_i}{1-p_i z^{-1}}}_{\text{filter stable poles}}$$

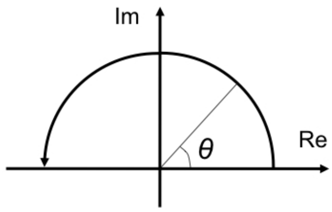
$$\Rightarrow y_k = G(e^{j\theta})e^{j\theta k} + \text{terms decaying to 0 as } k \to \infty.$$

$$\Rightarrow \text{Stationary output } \tilde{y}_k = G(e^{j\theta})e^{j\theta k} = |G(e^{j\theta})|e^{j(\theta k + \angle G(e^{j\theta}))}$$

Reminder: "Stationary response", "steady-state response" and "particular solution/integral" are synonyms.

Steady-state response to a *real* sinusoidal input





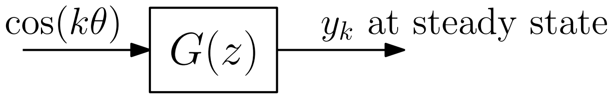
Use linearity

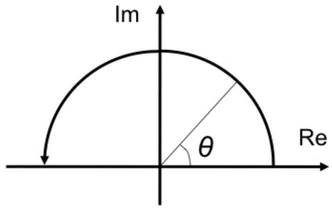
$$x_k = \cos(\theta k) = \frac{1}{2}\left(e^{j\theta k} + e^{-j\theta k}\right)$$

$$\tilde{y}_k = \frac{|G(e^{j\theta})|}{2}\left(e^{j(\theta k + \angle G(e^{j\theta}))} + e^{-j(\theta k + \angle G(e^{j\theta}))}\right)$$
$$= |G(e^{j\theta})| \cos(\theta k + \angle G(e^{j\theta}))$$

The filter attenuates the sinusoidal input $\cos(\theta k)$ by a factor $|G(e^{j\theta})|$ and shifts its phase by a factor $\angle G(e^{j\theta})$.

Frequency plots: Bode diagrams

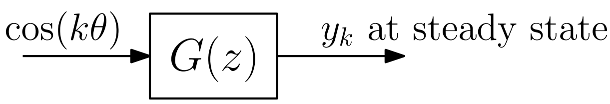


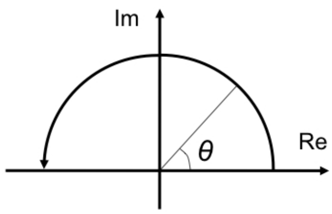


$$x_k = \cos(\theta k)$$
$$\tilde{y}_k = |G(e^{j\theta})| \cos(\theta k + \angle G(e^{j\theta}))$$

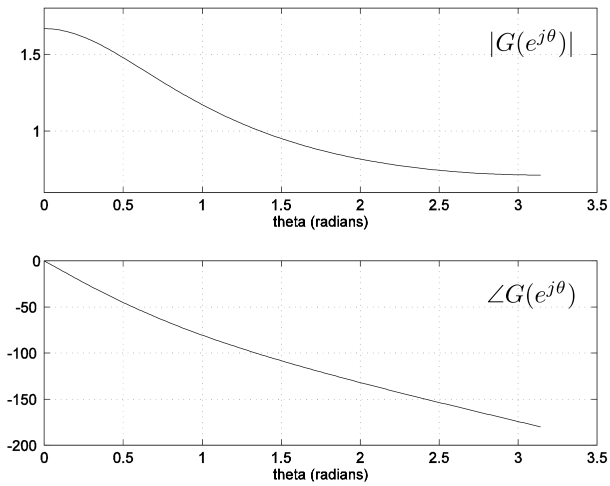
- Filter amplification and phase shift can be characterized at each frequency $\rightarrow$ Bode diagrams.
- Bode diagrams provide a good representation of the system behaviour since every signal at the input can be decomposed into sum of sinusoids.
- Note that $\cos(\theta k) = \cos((\theta + 2\pi)k)$ so going more than a complete revolution is redundant.
- Negative frequencies mirror positive frequencies for **real**-valued signals. Hence Bode diagrams often limited to the frequency interval $[0, \pi)$. If your signal is **complex**, use frequency interval $[-\pi, \pi)$ or equivalently $[0, 2\pi)$.

Bode diagram



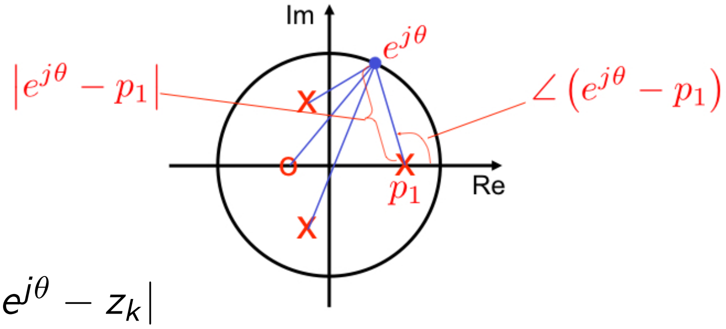


$$x_k = \cos(\theta k)$$
$$\tilde{y}_k = |G(e^{j\theta})| \cos(\theta k + \angle G(e^{j\theta}))$$



Interpreting a Bode diagram

- $G(z) = c\frac{\prod_{k=1}^{m}(z-z_k)}{\prod_{k=1}^{n}(z-p_k)}$

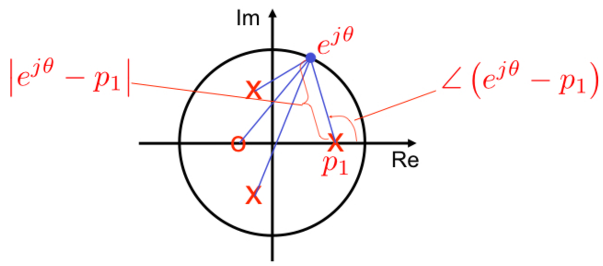


- $|G(e^{j\theta})| = |c|\frac{\prod_{k=1}^{m}|e^{j\theta}-z_k|}{\prod_{k=1}^{n}|e^{j\theta}-p_k|}$

$$|G(e^{j\theta})|_{dB} = 20\log(|G(e^{j\theta})|)$$
$$= 20\Big(\log|c| + \sum_{k=1}^{m}\log|e^{j\theta}-z_k| - \sum_{k=1}^{n}\log|e^{j\theta}-p_k|\Big)$$

- $\angle G(e^{j\theta}) = \angle(c) + \sum_{k=1}^{m}\angle(e^{j\theta}-z_k) - \sum_{k=1}^{n}\angle(e^{j\theta}-p_k)$

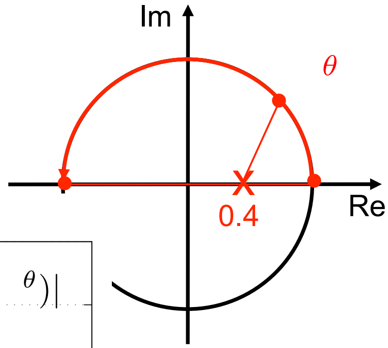
Hints on Bode diagrams

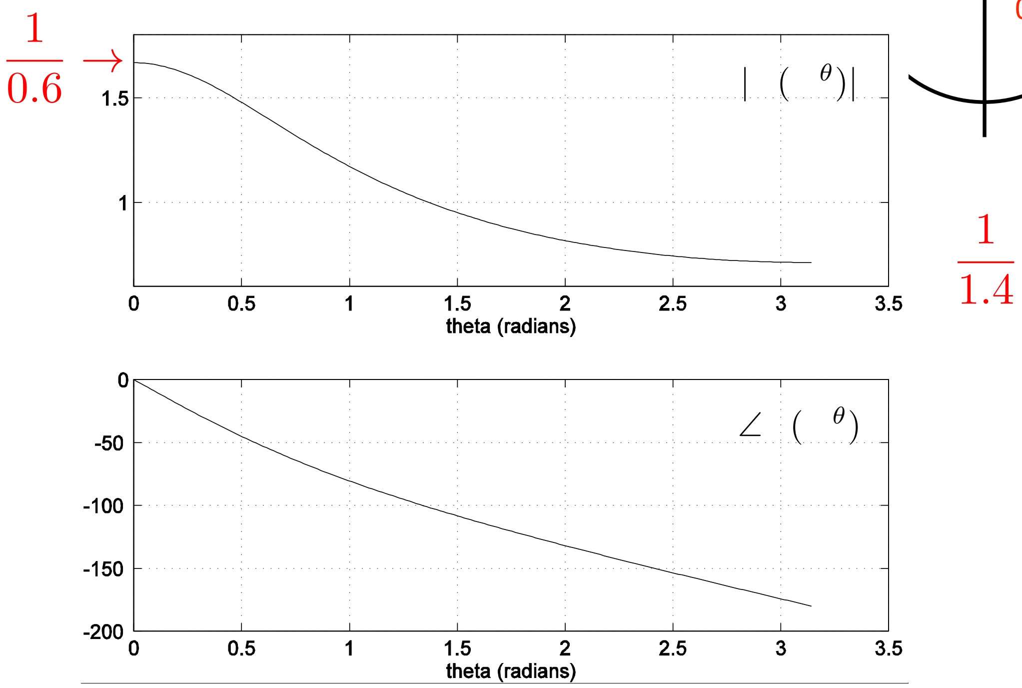


- The magnitude of the frequency response in dB is given by the sum of the log distances from the zeros to $e^{j\theta}$ minus the sum of the log distances from the poles to $e^{j\theta}$
- The phase response is given by the sum of the angles from the zeros to $e^{j\theta}$ minus the sum of the angles from the poles to $e^{j\theta}$.
- Thus when $e^{j\theta}$ "is close to" a pole, the magnitude of the response rises (resonance). When $e^{j\theta}$ "is close to" a zero, the magnitude falls (a null).
- The behaviour of the phase response is less intuitive but similar principles apply.
- Unlike the continuous case, there are no simple rules for drawing Bode diagrams of digital systems. Use numerical tools!

Bode diagrams in normalised frequencies

Example $\dfrac{1}{\quad - 0.4}$ which is a simple low pass filter.





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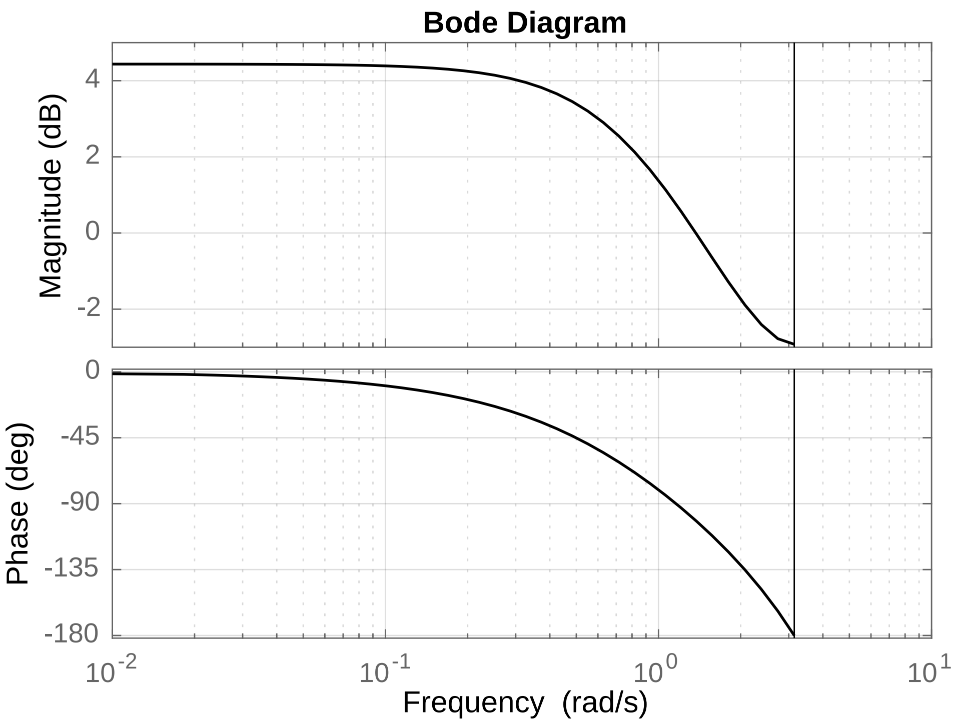
Bode diagrams in dB and rad s^{-1}



(Always better to use Hz than rad s^{-1})

To log or not to log, that is the question...

- frequencies for continuous-time Bode diagrams from 0 to ∞
  $\longrightarrow$ it always makes sense to plot against log frequencies
- for discrete-time systems, normalised frequencies from 0 to π, while actual frequencies go from 0 to $f_s/2$
  $\longrightarrow$ whether to plot against linear or logarithmic frequencies depends on the use case and what frequencies within the range you are most interested in
- if the salient features of your filter are at frequencies $\ll f_s$, plot against log frequencies
- otherwise use linear frequencies

Course outline

- you are now able to complete Examples Paper 1
- Next 2 lectures, we will learn how to stabilise an "open loop" system by using feedback and hence turning it into a stable "closed loop" system (Nyquist diagram)
- Thereafter we will leave control theory behind and concentrate on signal processing (filtering and filter design, discrete/analogue interfaces, DFT)