3F1 Signals and Systems: Handout 4

System transfer function & stability

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z-transfer function of an LTIS

A Linear Time-Invariant System (LTIS) described by linear difference equations in canonical form

$$y_k + a_1 y_{k-1} + ... + a_n y_{k-n} = b_0 x_k + ... + b_m x_{k-m}$$

and subject to zero initial conditions

$$y_k = x_k = 0 \text{ for } k < 0$$

gives in the z domain

$$Y(z) + a_1 z^{-1} Y(z) + ... + a_n z^{-n} Y(z) = b_0 X(z) + ... + b_m z^{-m} X(z)$$

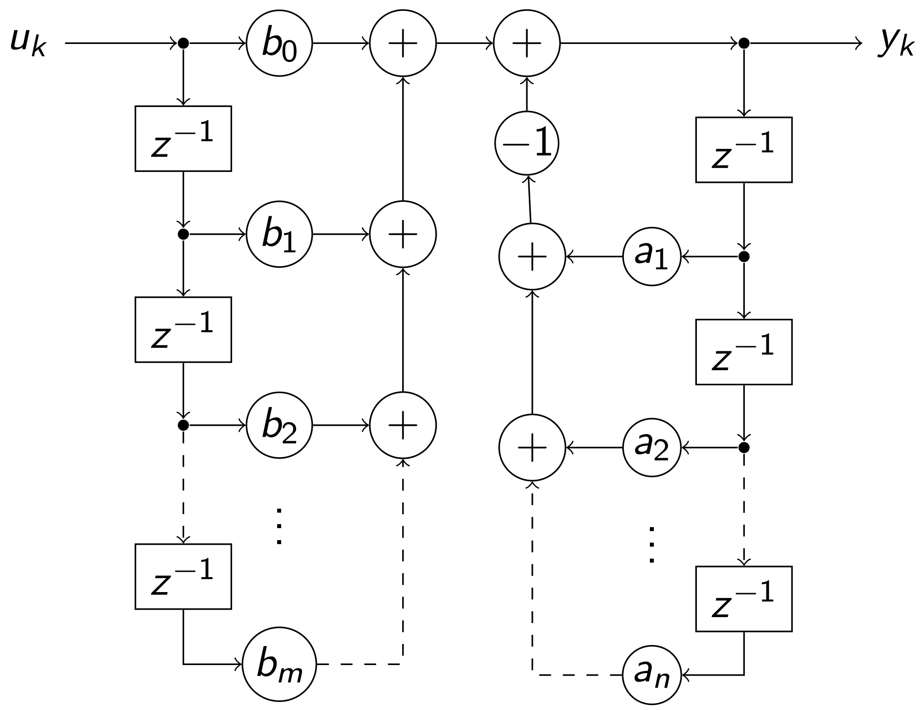
and is hence described by a **transfer function**:

$$G(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \ldots + b_m z^{-m}}{1 + a_1 z^{-1} + \ldots + a_n z^{-n}}$$

Note: ratio is independent of $X(z)$.

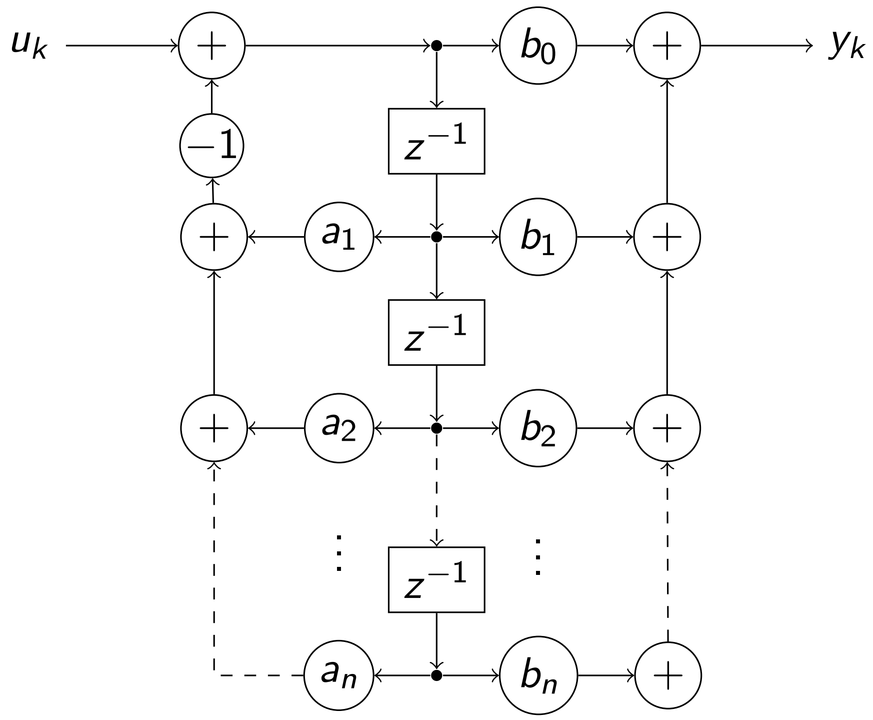
Direct Form I implementation of an LTIS

$$G(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_m z^{-m}}{1 + a_1 z^{-1} + \ldots + a_n z^{-n}}$$



Direct Form II implementation of an LTIS

$$G(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_n z^{-n}}{1 + a_1 z^{-1} + \ldots + a_n z^{-n}} \text{ where } b_k = 0 \text{ for } k = m{+}1, \ldots, n.$$



FIR/IIR in the z domain

- Direct form II is preferable. The two forms are equivalent because Direct Form I is a cascade of linear systems and you can invert the order of the two systems to obtain Direct Form II (multiplication is commutative)
- the a coefficients (denominator of $G(z)$) point in opposite direction from input to output: **feedback**
- the b coefficients (numerator of $G(z)$) point in the direction from input to output: **feed-forward**
- an **FIR** (finite impulse response) satisfies $a_n = 0$ for all n, i.e., $G(z)$ has only a numerator and a denominator of 1: it's a **feed-forward only system**. The output y_k only depends on current and past values of the input $x_k, x_{k-1}, \ldots$ but *not* on past values of the output $y_{k-1}, y_{k-2}, \ldots$
- an **IIR** (infinite impulse response) has an infinite length delta response as a result of **feedback**

Poles and Zeros

Switching from negative to positive powers of z, assuming $m < n$,

$$G(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_m z^{-m}}{1 + a_1 z^{-1} + \ldots + a_n z^{-n}} = \frac{b_0 z^n + b_1 z^{n-1} + \ldots + b_m z^{n-m}}{z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n}$$

or if $n < m$,

$$G(z) = \frac{b_0 z^m + b_1 z^{m-1} + \ldots + b_1 z + b_m}{z^n + a_1 z^{n-1} + \ldots + a_n z^{m-n}}$$

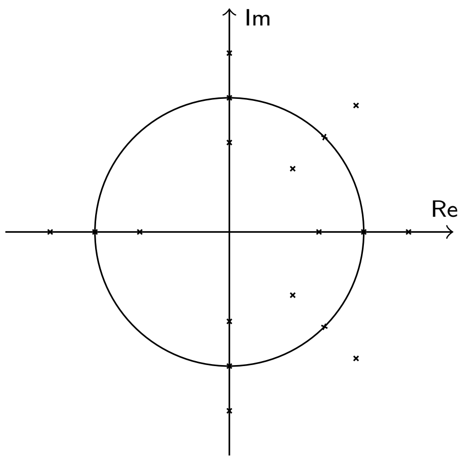
- An LTIS in canonical form corresponds to a rational function with equal degrees $\max\{m, n\}$ in numerator and denominator
- fundamental theorem of algebra $\longrightarrow$ every polynomial of degree n has n complex roots (counted with multiplicity, i.e., some of them could be co-located)
- a discrete LTIS has $\max\{m, n\}$ poles and $\max\{m, n\}$ zeros
- for an FIR, all the m poles are at $z = 0$

Delta response and poles

Consider a (LTI) system $G(z)$ subjected to a delta input: $\delta_k = (1, 0, 0, \dots) \longrightarrow \Delta(z) = 1$. What is the output?

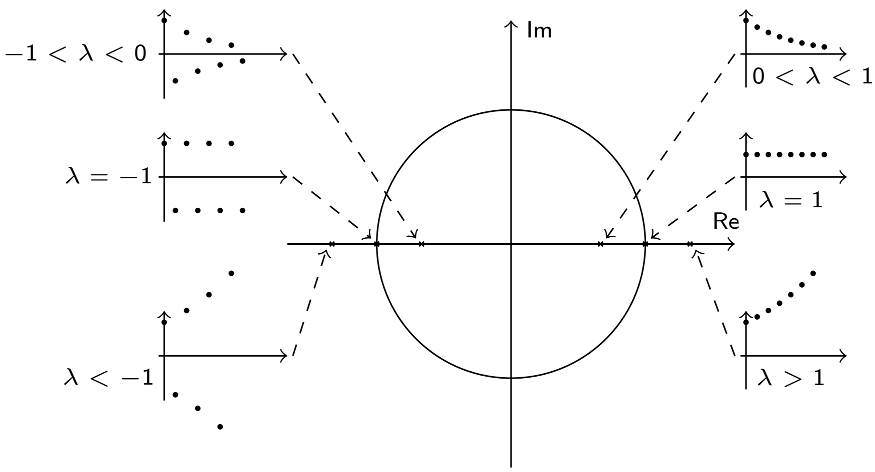
$$y = \mathcal{Z}^{-1}[Y(z)] = \mathcal{Z}^{-1}[G(z)\Delta(z)] = \mathcal{Z}^{-1}[G(z)]$$

$\Rightarrow$ poles of $G(z)$ define the response to a pulse



Real poles

$$G(z) = \frac{1}{1 - \lambda z^{-1}} \quad \xrightarrow{\mathcal{Z}^{-1}} \quad \lambda^k$$

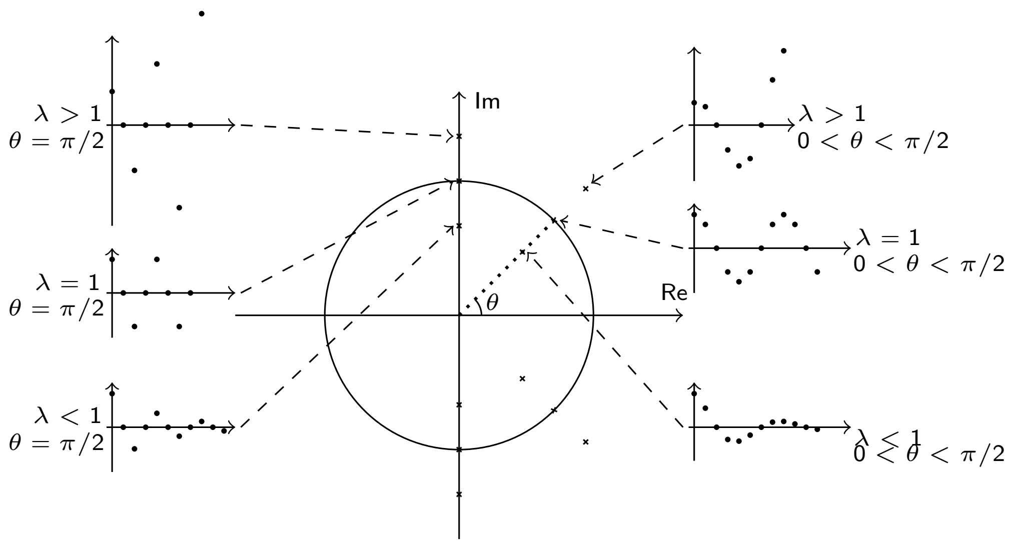


Complex conjugate poles

$$G(z) = \frac{1}{1-(\lambda e^{j\theta})z^{-1}} + \frac{1}{1-(\lambda e^{-j\theta})z^{-1}}$$

$$\xrightarrow{\mathcal{Z}^{-1}} \qquad \lambda^k\left(e^{j\theta k} + e^{-j\theta k}\right) = 2\lambda^k \cos(\theta k)$$

(real impulse response $\rightarrow$ pair of complex conjugate poles)



Repeated poles

- $G(z) = \frac{1}{(1-pz^{-1})^2}$

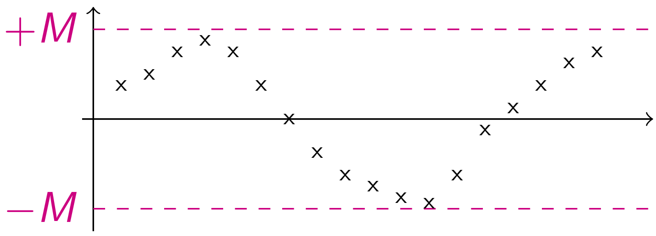
  rewrite as $G(z) = \frac{1}{1-pz^{-1}} + \frac{pz^{-1}}{(1-pz^{-1})^2}$ then $\xrightarrow{\mathcal{Z}^{-1}} p^k + kp^k$

Damping and Oscillation

- distance of poles from origin is a measure of decay rate
- complex poles just inside unit circle give lightly damped oscillation
- oscillation is possible for real poles on negative real axis

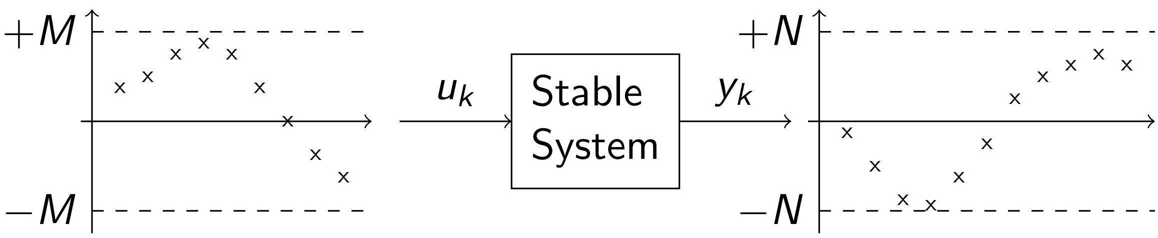
Bounded signals

We say that a signal $\{u_k\}$ is **bounded** if there exists a positive constant M such that $|u_k| < M$ for all k.



BIBO Stability

A discrete time system is **stable** if
<u>bounded inputs give bounded outputs</u> ("BIBO" stability).



> For any constant M, there exists a constant N such that, *for any* input signal bounded by M the output is bounded by N.

- The definition is more subtle than appears:
  - If a bounded input gives an unbounded output, the system is unstable
  - However, for some unstable systems there is no unbounded output but a collection of input signals bounded by M give outputs that cannot be bounded by any single N (See Examples Paper 1)

Theorem (Conditions for stability of a discrete time system)

Let G be a discrete time system with a rational transfer function,

$$G(z) = \frac{b(z)}{a(z)} = \frac{b_0 + \ldots + b_m z^{-m}}{1 + a_1 z^{-1} + \ldots + a_n z^{-n}}$$

with no common factors between $b(z)$ and $a(z)$. Let the pulse response of G be $\{g_k\}_{k\geq 0}$. Then the following are equivalent:

1. *G is stable*
2. *All of the roots p_i of $a(z)$ (i.e. poles) satisfy $|p_i| < 1$*
3. *$\sum_{k=0}^{\infty} |g_k|$ is finite*

Logical sequence of proof:

$$(1) \overset{A}{\Longrightarrow} (2) \overset{\text{Ex. paper}}{\Longrightarrow} (3) \overset{B}{\Longrightarrow} (1)$$

(Moreover, Example sheet shows $1 \implies 3$ and re-visits $1 \implies 2$ with real input signals .)

Reminder: logic and mathematical proofs

- $A \implies B$ means that if A is true, then B is true. The reverse doesn't necessarily hold: it is possible for B to be true but A to be false. For example
  - $x = 2 \implies x^2 = 4$
  - but $x^2 = 4 \not\Longrightarrow x = 2$ (x could be -2!)
- Implication is transitive: $A \implies B \implies C$ implies $A \implies C$
- Equivalence: $A \iff B$ means $A \implies B$ and $B \implies A$, also known as "if and only if": A is true if and only if B is true.
- To prove equivalence, you need to prove the two implications, i.e., $A \implies B$ and $B \implies A$
- $A \implies B$ is equivalent to $\neg B \implies \neg A$, where $\neg$ is the negation operator ("not B implies not A"). Often it is easier to prove $\neg B \implies \neg A$ than $A \implies B$, but since these statements are equivalent, the proof is equally valid.

Proof, part A

A: G stable $\implies$ all of the roots of $a(z)$ satisfy $|p_i| < 1$.

We will prove $\neg(|p_i| < 1 \text{ for all } i) \implies G$ not stable, i.e., if there is at least one root for which $|p_i| \geq 1$, then G is not stable.

Suppose $p_1, p_2, p_3, \ldots$ are distinct. Then we can decompose G using partial fractions:

$$G(z) = \frac{\alpha_1}{1 - p_1 z^{-1}} + \ldots + \frac{\alpha_n}{1 - p_n z^{-1}}$$

Then

$$g_k = \alpha_1 p_1^k + \alpha_2 p_2^k + \ldots + \alpha_n p_n^k$$

Suppose $|p_i| > 1$ for some i. Then g_k is unbounded.

Therefore **a delta input (bounded) gives an unbounded output and hence G is not stable.**

The proof with repeated poles is more involved but follows similar lines.

What about if there is no pole such that $|p_i| > 1$ but a pole with $|p_i| = 1$?

Pole on the unit circle

$$G(z) = \frac{\alpha_i}{1 - e^{j\theta} z^{-1}} + \tilde{G}(z)$$

where $\tilde{G}(z)$ contains the remaining terms in the partial fraction expansion of $G(z)$. Recall that $g_k = \alpha_1 p_1^k + \ldots + \alpha_n p_n^k$ and hence the i-th term in the sum corresponding to this pole is

$$g_k' = \alpha_i e^{j\theta k}$$

whose magnitude $|g_k'| = \alpha_i$ is constant for all times and hence bounded.

Now consider the input

$$U(z) = \frac{\alpha_i z^{-1}}{1 - e^{j\theta} z^{-1}}.$$

It corresponds to the sequence g_{k-1}' as it is simply the time-shift operator z^{-1} applied to the z transform of g_k', and hence it is bounded.

Pole on the unit circle (continued)

The correponding output is

$$Y(z) = U(z)G(z) = U(z)\left(\frac{\alpha_i}{1 - e^{j\theta}z^{-1}} + \tilde{G}(z)\right)$$
$$= \frac{\alpha_i^2 z^{-1}}{(1 - e^{j\theta}z^{-1})^2} + U(z)\tilde{G}(z)$$

The first term in this sum corresponds to the time sequence

$$y_k' = k\alpha_i^2 e^{j\theta(k-1)}$$

whose magnitude grows with k and is therefore unbounded! Hence we have found a bounded input g_{k-1}' that gives an output y_k with an unbounded component y_k'.

We have completed the proof that

$$G(z) = \frac{b(z)}{a(z)} \text{ is stable } \implies \text{ the roots of } a(z) \text{ satisfy } |p_i| < 1$$

.

Proof, part B

B: $\sum_{k=0}^{\infty} |g_k| < \infty \implies G$ stable (b. input $\implies$ b. output)

Suppose that $\sum_{k=0}^{\infty} |g_k| < \infty$, i.e., the sum is finite. Let $\{u_k\}$ be an aribtrary **bounded** input, i.e. $|u_k| < M$ for $k \geq 0$. Then the output, $\{y_k\}$ is given by the convolution

$$y_k = \sum_{i=0}^{k} g_i u_{k-i}$$

Thus,

$$|y_k| = \left|\sum_{i=0}^{k} g_i u_{k-i}\right|$$
$$\leq \sum_{i=0}^{k} |g_i|\,|u_{k-i}|$$
$$< \sum_{i=0}^{k} |g_i| M = M \sum_{i=0}^{k} |g_i| < \infty$$