3F1 Signals and Systems: Handout 3

z transform properties

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Michaelmas Term 2025

Initial Value Theorem

$$\lim_{z\to\infty} U(z) = \lim_{z\to\infty} \sum_{k=0}^{\infty} u_k z^{-k}$$
$$= \lim_{z\to\infty} \left(u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \dots \right)$$
$$= u_0$$

Conjugate symmetry property

- Like the Fourier transform, the z transform has a conjugate symmetry property. If a discre-time signal $\mathbf{s}$ is real,

$$U(\bar{z}) = \sum_{k=0}^{\infty} u_k \bar{z}^{-k} = \overline{\sum_{k=0}^{\infty} u_k z^{-k}} = \overline{S}(z)$$

- Consider for example the transform $G(z) = Az/(z-q)$ of the *real* geometric sequence $\mathbf{g} = \{Aq^k\}_{k\geq 0}$, with A and q real,

$$G(\bar{z}) = A\frac{\bar{z}}{\bar{z}-q} = A\frac{\bar{z}}{\overline{z-q}} = \overline{G(z)}$$

- On the unit circle (and for the DTFT), this results in

$$\begin{cases} |U(e^{-j\theta})| = |U(e^{j\theta})| \\ \angle U(e^{-j\theta}) = -\angle U(e^{j\theta}) \end{cases}$$

Reverse symmetry property

- The DTFT also has a reverse symmetry property: if

$$u_{-k} = \bar{u}_k,$$

then

$$U(\theta) = \sum_{-\infty}^{\infty} u_k e^{-j\theta k} = u_0 + \sum_{k=1}^{\infty}(u_{-k}e^{j\theta k} + u_k e^{-j\theta k})$$

$$= u_0 + \sum_{k=1}^{\infty}(\overline{u_k e^{-j\theta k}} + u_k e^{-j\theta k})$$

$$= u_0 + 2\sum_{k=1}^{\infty} \mathrm{Re}(u_k e^{-j\theta k})$$

i.e., the DTFT $U(\theta)$ is real

- For example, consider the signal $(u_{-1}, u_0, u_1) = (-j, -1, j)$,

$$U(\theta) = -je^{j\theta} - 1 + je^{-j\theta} = 2\sin\theta - 1$$

- there is no reverse symmetry property for the (one-sided) z transform because it doesn't depend on signal values for $k < 0$

Inverse z transform

There are several methods to invert the z transform.

1. Use partial fractions to express the function as a sum of known z transforms
2. Use polynomial division to calculate the power series and hence the time samples
3. Use the Taylor expansion of a function to get the power series and hence the time samples
4. Using the *residue theorem* of complex analysis (not taught in the Cambridge Engineering Tripos)

The DTFT can be inverted with all the methods above, or using the DTFT inversion formula derived in 2P6 Examples Paper 7, Question 5

$$u_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} U(\theta)e^{jk\theta}d\theta$$

Example: inverting a Z-transformed signal

$$U(z) = \frac{z-3}{z^2(z-1)(z-2)}$$

What is $\{u_k\}$, $k \geq 0$?

$$\begin{aligned} U(z) &= \frac{z(1-3z^{-1})}{z^4(1-z^{-1})(1-2z^{-1})} \\ &= z^{-3}\left(\frac{A}{1-z^{-1}} + \frac{B}{1-2z^{-1}}\right) \\ &= \underbrace{z^{-3}}_{\text{delay by 3}}\left(\underbrace{\frac{2}{1-z^{-1}}}_{2\{1^k\}_{k\geq 0}} + \underbrace{\frac{-1}{1-2z^{-1}}}_{-\{2^k\}_{k\geq 0}}\right) \end{aligned}$$

Therefore,

$$u_k = \begin{cases} 0, & k = 0,1,2 \\ 2-2^{k-3}, & k \geq 3 \end{cases}$$

Remark: we can use **polynomial long division** to explicitly write down the first few terms of $\{x_k\}$. In the previous example:

$$U(z) = \frac{z-3}{z^2(z-1)(z-2)} = \frac{z-3}{z^2(z^2-3z+2)}$$

$$\begin{array}{r l}
 & z^{-1} - 2z^{-3} - 6z^{-4} \ldots \\
z^2 - 3z + 2) & z - 3 \\
 & -(z - 3 + 2z^{-1}) \\
 & \quad - 2z^{-1} \\
 & \quad -(-2z^{-1} + 6z^{-2} - 4z^{-3}) \\
 & \quad\quad - 6z^{-2} + 4z^{-3}
\end{array}$$

$$\begin{aligned} U(z) &= z^{-2}(z^{-1} - 2z^{-3} - 6z^{-4} + \ldots) \\ &= z^{-3} - 2z^{-5} - 6z^{-6} + \ldots \\ \{u_k\}_{k\geq 0} &= (0, 0, 0, 1, 0, -2, -6, \ldots) \end{aligned}$$

Long division recovers a few numerical samples, whereas partial fractions recovers an expression for all k

The DTFT and Fourier series

- DTFT:

$$U(\theta) = \sum_{k=-\infty}^{\infty} u_k e^{-j\theta k} \quad u_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} U(\theta) e^{jk\theta} d\theta$$

- Complex Fourier Series (for $T = 2\pi$):

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jkt} \quad c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) e^{-jkt}$$

- Fourier series were not presented as a transform but they have all the properties of a transform, mapping a continuous periodic function onto a discrete set of coefficients
- Fourier series have many of the same properties we derived in this handout: conjugate symmetry, shift properties, convolution property for the periodic convolution $g \star f = \frac{1}{T}\int_0^T g(\tau) f(t-\tau) d\tau$
- Fourier series $\iff$ DTFT (or z transform on unit circle) with time and frequency swapped

The Laplace transform and the z transform

Applying the Laplace transform to a "train of impulses" sampled signal

$$w_s(t) = \sum_{k=0}^{\infty} w(kT)\delta(t - kT)$$

we obtain

$$W_s(s) = \int_0^{\infty} \sum_{k=0}^{\infty} w(kT)\delta(t - kT)e^{-st}dt$$
$$= \sum_{k=0}^{\infty} w(kT)e^{-skT} = \sum_{k=0}^{\infty} w_k z^{-k}$$

The z transform is equivalent to the Laplace transform of the sampled "impulse train" signal using a mapping from the Laplace to the z: domain

$$s \longrightarrow z = e^{sT}$$

Solution of linear difference equations

Example: a discrete time system is governed by the difference equation:

$$y_{k+2} = y_{k+1} - 0.25y_k + u_k$$

with initial conditions $y_0 = 1, y_1 = 0$. Find the step response.

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z transform of a step function: $u_k = 1^k$ for $k \geq 0$ is

$$U(z) = \frac{1}{1 - z^{-1}}$$

From data book: $\mathcal{Z}[\{y_{k+m}\}] = z^m Y(z) - z^m y_0 - ... - zy_{m-1}$
We can now apply the z tranform to the difference equation to yield

$$z^2Y(z) - z^2 = zY(z) - z - 0.25Y(z) + \frac{1}{1 - z^{-1}}$$

where we used the z transforms

$$\begin{cases} y_{k+2} \longrightarrow z^2Y(z) - z^2y_0 - zy_1 \\ y_{k+1} \longrightarrow zY(z) - zy_0 \end{cases}$$

Solving linear difference equations (continued)

Rearranging, we obtain

$$Y(z) = \frac{1 - 2z^{-1} + 2z^{-2}}{1 - 2z^{-1} + 1.25z^{-2} - 0.25z^{-3}} = \frac{1 - 2z^{-1} + 2z^{-2}}{\left(1 - z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)^2}.$$

then applying partial fractions

$$Y(z) = \frac{4}{1 - z^{-1}} + \frac{2}{1 - \frac{1}{2}z^{-1}} - \frac{5}{\left(1 - \frac{1}{2}z^{-2}\right)^2}$$

and hence returning to the time domain

$$y_k = 4 + 2\left(\frac{1}{2}\right)^k - 10(k+1)\left(\frac{1}{2}\right)^{k+1} \quad \text{for } k \geq 0$$

where we rewrote

$$\frac{5}{\left(1 - \frac{1}{2}z^{-1}\right)^2} = 10z \frac{\frac{1}{2}z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)^2}$$

Course outline

- We have completed the properties of z transform and DTFT
- You can attempt questions 1-5 in Examples Paper 1
- Next 4 lectures we will look at system properties such as
  1. system transfer function
  2. system stability
  3. frequency response (Bode diagrams)
  4. stability of closed-loop systems (Nyquist diagrams)