3F1 Signals and Systems: Handout 2

The z transform aand the DTFT

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Introduction

- z transform is the discrete-time equivalent of the Laplace transform
- DTFT is the discrete-time equivalent of the Fourier transform

More fundamentally,

- why do transforms work?
- in discrete-time, transforms are much easier to motivate than in continuous time...
- we will begin by motivating discrete-time transforms in general (not specifically z transform or DTFT)
- the motivation extends naturally to all transforms

Convolution and Polyomial Multiplication

- Consider a convolution of two finite length signals $c = (c_0, c_1)$ and $u = (u_0, u_1, u_2)$
- Let's compute $y = c \star u$ using $y_k = \sum_{\ell=0}^{1} c_\ell u_{k-\ell}$:

$$\begin{cases} y_0 = c_0 u_0 \\ y_1 = c_0 u_1 + c_1 u_0 \\ y_2 = c_0 u_2 + c_1 u_1 \\ y_3 = c_1 u_2 \\ y_k = 0 \quad \text{for } k < 0 \text{ and for } k > 3 \end{cases}$$

- This bears an uncanny resemblance with **polynomial multiplication**

Convolution and Polyomial Multiplication (continued)

- Define two polyomials

$$\begin{cases} c(D) = c_0 + c_1 D \\ u(D) = u_0 + u_1 D + u_2 D^2 \end{cases}$$

- We used the variable name **D** to stand for **"Dummy"** to emphasise the fact that, for the moment, we attach no significance to the argument of the polynomial. We are only using the polynomial representation as a formal help to define convolution
- Let us now compute the polynomial multiplication:

$$y(D) = c(D)u(D) = c_0 u_0 + (c_0 u_1 + c_1 u_0)D + (c_0 u_2 + c_1 u_1)D^2 + c_1 u_2 D^3$$

- We see that the coefficients of $y(D)$ are indeed the outcomes (y_0, y_1, y_2, y_3) of the convolution $c \star u$!
- Conclusion (for this simple example): **convolution of sequences is equivalent to multiplication of polynomials**

Convolution and power series

We can use the same trick for **semi-infinite sequences** by mapping them to **power series**

$$u(D) = u_0 + u_1 D + u_2 D^2 + u_3 D^3 \dots$$

Then

$$\begin{aligned} c(D)u(D) &= \left(\sum_{k=0}^{\infty} c_k D^k\right)\left(\sum_{m=0}^{\infty} u_m D^m\right) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} c_k u_m D^{k+m} \\ &= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} c_k u_{n-k} D^n \text{ where we substituted } n = k + m \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} c_k u_{n-k}\right) D^n \quad \text{(see separate slide for justification)} \\ &= \sum_{n=0}^{\infty} y_n D^n \text{ where } y_n = \sum_{k=0}^{n} c_k u_{n-k}, \text{ i.e., } y = c \star u \end{aligned}$$

Justification for swapping $\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}$ with $\sum_{n=0}^{\infty}\sum_{k=0}^{n}$

Let $a_{k,n} = c_k u_{n-k} D^n$, then

$$\begin{array}{llllll} \to n & & & & & \\ \downarrow & a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} & \dots \\ k & & a_{1,1} & a_{1,2} & a_{1,3} & \dots \\ & & & a_{2,2} & a_{2,3} & \dots \\ & & & & a_{3,3} & \dots \end{array}$$

The first expression sums row-wise, the second expression sums column-wise.

Convolution-multiplication equivalence

- Discrete **convolution** of finite signals is equivalent to **multiplication** of polynomials
- Discrete **convolution** of semi-infinite signals is equivalent to **multipication** of power series

- By converting signals to polynomials/power series, we replace **convolution** by **multiplication**
- All discrete transforms use this trick
- All discrete transforms have the **"convolution property"**
- Continuous transforms can be seen as an extension of this principle, where power series become "functions" (of s or $j\omega$) and sums become integrals

Representing sequences as power series

- Swapping sequences for power series is a common trick used throughout engineering and mathematics:
  - **Generating functions** in combinatorics and other branches of mathematics (e.g. the **Probability Generating Function (PGF)** for discrete random variables)
  - D-**transform** in the study of error correction "convolutional" codes (see 3F4 & 4F5)
  - the z **tranform** and the DTFT in digital control and digital signal processing
- In all those, the power series can be seen as a substitute representation of the sequence, hence the term "transform"
- From the Wikipedia article on the generating function:

  A generating function is a device somewhat similar to a bag. Instead of carrying many little objects detachedly, which could be embarrassing, we put them all in a bag, and then we have only one object to carry, the bag. - George Pólya, Mathematics and plausible reasoning (1954)

Power Series vs. Formal Power Series

- A formal power series uses a formal variable, e.g., D, where we never assign a value to the variable.
- Formal power series are a way to harness computation rules for power series. For example, in formal power series
$$\frac{1}{1-D} = 1 + D + D^2 + D^3 \dots$$
always holds, resulting from the rules of polynomial division.
- For (non-formal) power series, the above is only true if $|D| < 1$ as the sum doesn't converge for $|D| \geq 1$.
- To map convolutions to multiplications, formal power series are sufficient and there is no need to assign values to the dummy variable or to worry about convergence. This is the approach for generating functions and the D transform.
- In the z transform/DTFT, we do assign a value to the variable $z/e^{j\theta}$ and hence it is a power series, not a formal power series. This is because these transforms have other interesting properties beyond converting convolutions to multiplications, such as indicating stability and determining steady state responses of signals.

The z transform

For a signal $\{u_k\}$, the z transform is defined as
$$U(z) = \sum_{k=0}^{\infty} u_k z^{-k}$$

- This is equivalent to our generic transform with $D = z^{-1}$. The negative power is adopted by analogy with the Laplace transform
- Note that we do not assume that $\{u_k\}$ starts at $k = 0$, but the z transform *ignores* all signal values at negative times
- The signal $\{u_k\}$ is real-valued in most control theory applications but can be complex in other applications
- For example, signal processing in a wireless device (e.g. your mobile phone, your Wi-Fi router) routinely uses the z transform applied to complex signals.
- Standard z transform and properties are listed in a table the *Information Data Book* on page 3.

Examples

- Consider $\mathbf{u} = \{u_k\}_{k\geq 0} = 1, 2, 3, 4, 5, 0, 0, 0, 0, \ldots$.

$$U(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-5}$$

- Consider the geometric sequence $\mathbf{u} = \{u_k\}_{k\geq 0} = \{1, q, q^2, q^3, \ldots\}$.

$$U(z) = \sum_{k=0}^{\infty} q^k z^{-k} = \sum_{k=0}^{\infty} (qz^{-1})^k = \frac{1}{1 - qz^{-1}} = \frac{z}{z - q}$$

  This expression is only valid provided $|qz^{-1}| < 1$ or $|z| > |q|$, in other words outside the circle of radius $|q|$. This is the region of convergence (ROC) of the z transform.

- Special case $q = 1$ is the unit step signal $\mathbf{u} = 1, 1, 1, \ldots$, i.e., $u_k = 0$ for all $k < 0$ and $u_k = 1$ for all $k \geq 0$, we have

$$U(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$$

Region of convergence (ROC)

Some textbooks consider the z tranform as:

$$\text{signal } \{u_0, u_1, u_2, \ldots\} \longleftrightarrow (\text{expression } U(z), \text{ a region of convergence})$$

- This is not strictly necessary and we will just say that $U(z)$ is the z tansform of $\{u_0, u_1, u_2, \ldots\}$
- In complex analysis, the concept of *analytic continuation* teaches that any "well behaved" function (e.g. a rational function such as $U(z) = \frac{z}{z-q}$) defined over a sub-region of the complex plane $\mathbb{C}$ has a unique continuation over the rest of $\mathbb{C}$. This is the basis on which we choose to ignore the ROC.
- Complex analysis is not taught in the Cambridge Engineering tripos

One-sided vs. two-sided

- The vast majority of textbooks define the z transform as we did
- A few textbooks define the z transform using a 2-sided sum from $-\infty$ to ∞
- This is mainly relevant for stochastic signals analysis
- This is delicate, because for example

$$\{u_k\}_{k\geq 0} = \{1, 1, 1, \ldots\} \longleftrightarrow U(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}$$

$$\{v_k\}_{k<0} = \{\ldots, 1, 1, 1\} \longleftrightarrow V(z) = \sum_{k=-\infty}^{-1} z^{-k} = \sum_{k=1}^{\infty} z^k = \frac{z}{z-1}$$

- The two signals have the *same* two-sided z transform!
- The two z transforms are only distinguised via their ROC in this case: $|z| > 1$ for $U(z)$, $|z| < 1$ for $V(z)$. For two-sided z transforms, one *cannot* ignore the ROC.
- We will never use the two-sided z transform: our two-sided transform is the DTFT

The Discrete-Time Fourier Transform (DTFT)

For a signal $\{u_k\}$, the DTFT is defined as

$$U(\theta) = \sum_{k=-\infty}^{\infty} u_k e^{-jk\theta}$$

- The DTFT was introduced in 2P6 Signal and Data Analysis
- We will study its properties, which are in general identical to the z transform properties, and in some cases simpler (e.g. time shift properties)
- If a signal is zero for negative values, e.g., $\{u_k\}_{k\geq 0}$, then

$$U(\theta) = U(z) \text{ for } z = e^{j\theta}, i.e., U(\theta) = U(e^{j\theta})$$

- For signals that are zero at non-negative values, the DTFT is equivalent to the z transform on the unit circle

Linearity of the z transform and DTFT

For any scalars α, β:

$$\mathcal{Z}[\alpha\{x_k\} + \beta\{y_k\}] = \alpha\mathcal{Z}[\{x_k\}] + \beta\mathcal{Z}[\{y_k\}]$$

This is because by definition, taking the z transform requires only linear operations on the signal values.

From this it also follows that, if the signal definition involves any parameter α,

$$\mathcal{Z}\left[\frac{d}{d\alpha}u_k\right] = \frac{d}{d\alpha}U(z)$$

The DTFT obeys the same properties.

More examples

- So far we have learned $\mathcal{Z}\{q^k\}_{k\geq 0} = \frac{1}{1-qz^{-1}}$
- We can use the differentiation property of linearity to obtain

  $$\mathcal{Z}\left[\frac{d}{dq}[1, q, q^2, \ldots]\right] = \mathcal{Z}[0, 1, 2q, 3q^2, \ldots] = \frac{d}{dq}\frac{1}{1-qz^{-1}} = \frac{z^{-1}}{(1-qz^{-1})^2}$$

  In other words, the z transform of the signal $u_0 = 0$, $u_k = kq^{k-1}$ for $k > 0$ is $\frac{z^{-1}}{(1-qz^{-1})^2}$.
- Special case: the z transform of the ramp sequence $[0, 1, 2, 3, \ldots]$, i.e., $u_k = k$ for $k \geq 0$ is: $\frac{z^{-1}}{(1-z^{-1})^2}$.
- The differentiation trick can be used repeatedly to obtain

  $$\mathcal{Z}\left[\frac{(k+m-2)!}{(k-1)!(m-1)!}q^k\right] = \frac{z^{-1}}{(1-qz^{-1})^m} \text{ for all } m \geq 1$$

More examples and properties

- the linearity properties can be applied easily to determine the z transforms of

$$\begin{aligned}\mathcal{Z}[\cos \alpha k] &= \mathcal{Z}[\frac{1}{2}(e^{j\alpha k} + e^{-j\alpha k})] \\ &= \frac{1}{2}\left(\frac{1}{1 - e^{j\alpha}z^{-1}} + \frac{1}{1 - e^{-j\alpha}z^{-1}}\right) \\ &= \frac{1}{2}\frac{1 - e^{-j\alpha}z^{-1} + 1 - e^{j\alpha}}{(1 - e^{j\alpha})(1 - e^{-j\alpha})} \\ &= \frac{1 - z^{-1}\cos\alpha}{1 - 2z^{-1}\cos\alpha + z^{-2}}\end{aligned}$$

- Similarly,

$$\mathcal{Z}[\sin \alpha k] = \frac{z^{-1}\sin\alpha}{1 - 2z^{-1}\cos\alpha + z^{-2}}$$

- Finally, it is easy to show using the definition of the z transform that scaling a sequence with a geometric sequence results in

$$\mathcal{Z}[r^k u_k] = \sum_{k=0}^{\infty} u_k r^k z^{-k} = \sum_{r=0}^{\infty} u_k (r^{-1}z)^{-k} = U(r^{-1}z)$$

Time shift properties of the DTFT

- Consider a sequence $\{u_k\}$ shifted by a "delay" d: $\mathbf{u}' = \{u_{k-d}\}$
- d can be negative in which case it's an "advance"
- The DTFT is

$$U'(\theta) = \sum_{k=-\infty}^{\infty} u_{k-d}e^{-jk\theta} = \sum_{k'=-\infty}^{\infty} u_{k'}e^{-j(k'+d)\theta} = e^{-jd\theta}U(\theta)$$

where we used the variable substitution $k' = k - d$

- Time shift for DTFT results simply in a multiplication by $e^{-j\theta d}$

Extending DTFT time-shift properties to the z transform

- We have learned that DTFT time shift results in a multiplication by $e^{-j\theta d}$, and we know that $U(\theta) = U(z)$ for $z = e^{j\theta}$ for signals that are zero at negative values
- It would be tempting to assume that the property is extended to the z transform as a multiplication by z^{-d}
- Unfortunately, things are trickier for the z transform due to its one-sided definition:
  - when shifting by a positive d ("delaying"), signal values that were formerly ignored because they were at negative times now appear in the z transform (this does not happen if $u_k = 0$ for $k < 0$)
  - when shifting by a negative d ("advancing") signal values that were formerly at non-negative times and included in the z transform must now be ignored because they end up at negative times

Time shift properties

1. **Time delay** $\{u'_k\} = \{u_{k-d}\}$

$$
\begin{aligned}
\mathcal{Z}[\{u_{k-d}\}] &= \sum_{k=0}^{\infty} u_{k-d} z^{-k} \\
&= u_{-d} + u_{-(d-1)} z^{-1} + \ldots + u_{-1} z^{-(d-1)} + \sum_{k=d}^{\infty} u_{k-d} z^{-k} \\
&= u_{-d} + u_{-(d-1)} z^{-1} + \ldots + u_{-1} z^{-(d-1)} + \sum_{k'=0}^{\infty} u_{k'} z^{-(k'+d)} \\
&= u_{-d} + u_{-(d-1)} z^{-1} + \ldots + u_{-1} z^{-(d-1)} + z^{-d} U(z)
\end{aligned}
$$

For signals $\{u_k\}_{k\geq 0}$ that are zero at negative values,

$$\mathcal{Z}[\{u_{k-d}\}] = z^{-d} \mathcal{Z}[\{u_k\}]$$

z^{-1} is therefore the **time-delay operator.**

Time shift properties (continued)

2. **Time advance** $\{u'_k\}\{u_{k+d}\}$

$$\begin{aligned}\mathcal{Z}[\{u_{k+d}\}] &= \sum_{k=0}^{\infty} u_{k+d} z^{-k} \\ &= \sum_{k'=d}^{\infty} u_{k'} z^{-(k'-d)} \\ &= \sum_{k'=0}^{\infty} u_{k'} z^{-(k'-d)} - \sum_{k'=0}^{d-1} u_{k'} z^{-(k'-d)} \\ &= z^d U(z) - z^d u_0 - z^{d-1} u_1 - \ldots - z u_{d-1}\end{aligned}$$

z is the **time-advance operator.**

- Using the time shift properties, one can adjust the expression obtained in the lecture for various sequences to obtain the expressions in the *Information Data Book*.
- The time-shift properties can be applied repeatedly to obtain the expressions in the data book for a delay by m or an advance by m.

Initial Value Theorem

$$\begin{aligned}\lim_{z\to\infty} U(z) &= \lim_{z\to\infty} \sum_{k=0}^{\infty} u_k z^{-k} \\ &= \lim_{z\to\infty} \left(u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \ldots\right) \\ &= u_0\end{aligned}$$

Convolution theorem (reminder)

Discrete convolution:

$$\mathbf{a} \star \mathbf{b} = \sum_{i=-\infty}^{\infty} a_i b_{k-i} = \sum_{i=-\infty}^{\infty} a_{k-i} b_i$$

The convolution theorem for Z-transforms:

$$\mathcal{Z}[\mathbf{a} \star \mathbf{b}] = A(z)B(z)$$

Remember: this is why we "invented" the z transform in the first place. See proof for all formal series based transforms in Slide 5.

Properties of z transforms

We have learned the following properties:

- Convolution property
- Linearity
- Scaling with e geometric sequence
- Time shift
- initial value theorem

Next lecture, we we learn one more property (conjugate symmetry), learn how to invert the z transform, how to use it to solve difference equations, and how it relates to continuous transforms