# 3F1 Signals and Systems: Handout 1

## Introduction, discrete time signals and systems

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# Introduction

This course is mostly about:

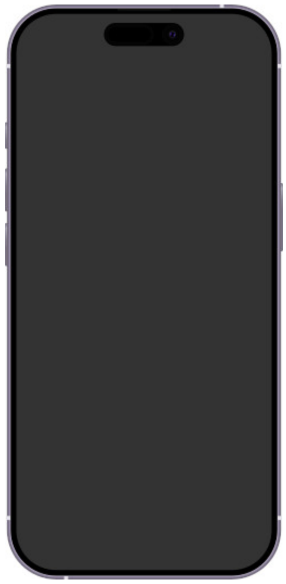
- **digital/discrete-time signals**
- **digital/discrete-time systems**

Why should we consider **discrete time systems**?

# Analogue or Digital?

Motorola DynaTAC
1983-94

Nokia 8110
1996



Apple iPhone 17
2025

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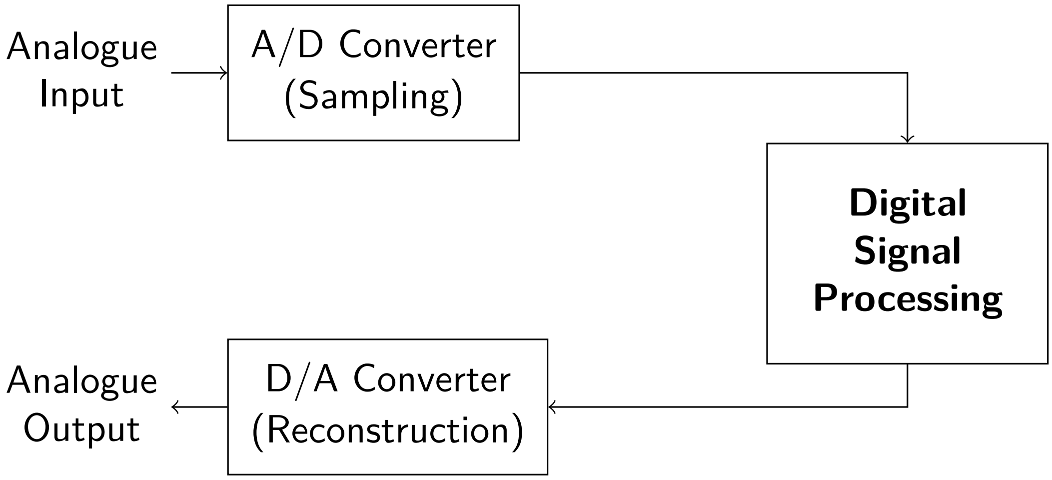
# What is digital?

Analogue Input → A/D Converter (Sampling) → D/A Converter (Reconstruction) → Analogue Output

- Sampling theorem states conditions for perfect reconstruction
- Input = Output, ☺
- But what's the point???

# Motivation




- Sampling theorem states conditions for which no information is lost between analogue and digital
- The point of switching to digital is that digital signal processing is far more powerful than analogue signal processing.



# Digital Signal Processing

| | |
|---|---|
| 3F1 | ▶ Digital filtering (to emphasise certain frequencies and attenuate others)<br>▶ Digital control (to stabilise a system) |
| 3F3 | ▶ Digital estimation, prediction or detection |
| 3F7 | ▶ Data compression<br>▶ Error correction |

# Course overview

- Discrete-time systems, Digital Filtering and the Discrete Fourier Transform (13 lectures)
    - Recap of continuous systems
    - the $z$ transform
    - discrete convolution
    - Bode diagrams
    - stability
    - continuous/discrete interfaces
    - FIR, IIR
    - design of filters
    - the Discrete Fourier Transform and Fast Fourier Transform
- Continuous time random signals (3 lectures)

The "continuous time random signals" part of the course is by Prof G Vinnicombe.



# Pre-requisites and background

| Topic | Source |
|---|---|
| Linear ordinary differential equations (ODEs) with constant coefficients | A-level Further Maths, revised in 1P4 MT |
| Linear difference equations | 1P4 MT |
| Convolution and impulse responses | 1P4 LT |
| Laplace Transform | 1P4 ET |
| Bode diagrams and Nyquist stability criterion | 2P6 MT |
| Sampling theorem $\star$ | 2P6 LT |
| Qantisation $\star$ | 2P6 LT |
| Fourier series $\star$ | 1P4 LT |
| Fourier transform $\star$ | 2P6 LT |
| Discrete Fourier transform (DFT) $\star$ | 2P7 LT but will be re-taught from scractch in 3F1 |
| Probability | 2P7 LT |

# Continuous-time signals and systems

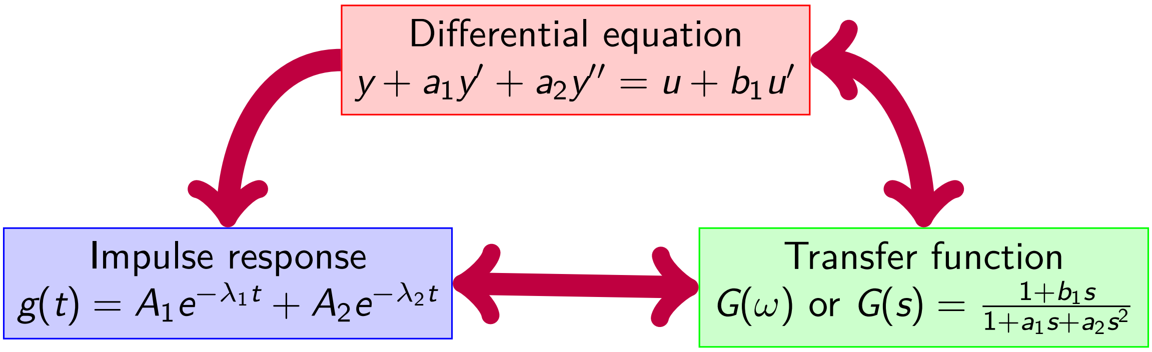



Figure: The 3 equivalent descriptions of a continuous-time LTIS

Tools:

- Bode diagrams (stationary response to sinusoidals)
- Nyquist diagrams (stability of closed-loop system)



# Digital vs. discrete-time

- **Digital:** discrete in time and amplitude
- Continuous-time and amplitude signals are *sampled* and *quantised*
- There are few theoretical tools for the analysis of full digital signal processing
- **Discrete-time:** discrete in time only
- Sampled but *not* quantised
- We will focus on **discrete-time signal processing**

## Continuous $\longrightarrow$ Discrete time

| | Contiuous time | Discrete time |
|---|---|---|
| Signals | Functions $s(t)$ of $t$ | **Sequences** $s_k$ indexed in $k$ |
| Systems | Differential equations | **Difference** equations |



## Discrete time signals

- Discrete time signals: $\mathbf{s} = (\ldots, s_{-2}, s_{-1}, s_0, s_1, s_2, \ldots)$
- The $x_k$ are typically real or complex numbers (not always, e.g., if you take 4F5 next year, they may be elements of so-called *finite fields*)
- **Right-sided signals:** $\mathbf{s} = (s_m, s_{m+1}, s_{m+2}, \ldots)$. $s_k = 0$ for $k < m$.
- **Left-sided signals:** $s_k = 0$ for $k > m$.
- Two-sided signals: go both directions
- **Finite-length signal:** $\mathbf{s} = (s_m, s_{m+1}, \ldots, s_{m+\ell-1})$.
- In this course we will focus mainly but not exclusively on **right-sided signals.**
- A right-sided signals starting at zero

$$\mathbf{s} = \{s_k\}_{k\geq 0} = s_0, s_1, s_2, \ldots$$

is also called a **semi-infinite sequence**

## Normalised sampling

- Discrete time signal or semi-infinite sequence $\{x_k\}_{k\geq 0}$ could be a sampled continuous signal, in which case, with reference to the original continuous sample,

$$s_k = s(kT)$$

  where $T$ is the sampling interval
- Signal processing deals with physical quantities: $s_k$ has a unit, $T$ is in seconds
- Signal processing techniques are independent of unit or sampling interval, so it's fair to treat a signal $\{s_k\}$ as a sequence of numbers
- Normalised time/frequency: $\tau = 1$ second, sampling frequency $\varphi_s = 1$ Herz, or $\theta_s = 2\pi$ radians per second
- When discussing a normalised frequency $\theta$, the true frequency can always be recovered as

$$f = \frac{\theta}{2\pi} f_s$$



## Discrete Time Systems

- A **discrete time system** takes discrete time signal inputs and produces discrete time signal outputs.
- The definition of a **Linear Time-Invariant (LTI)** system $L$ extends naturally from continuous to discrete time:
  - If $y_k$ is the output for the input sequence $u_k$ and $y'_k$ to $u'_k$ for any input sequences $u$ and $u'$,
  - **Linear:** $L(\alpha_1 u_k + \alpha_2 u'_k) = \alpha_1 y_k + \alpha_2 y'_k$
  - **Time-invariant:** $L(u_{k+m}) = y_{k+m}$ for any $m$.
- Continuous LTI systems are governed by linear **differential** equations with constant coefficients, e.g., $y + \alpha y' + \beta y'' = u(t)$
- Similarly, discrete LTI systems are governed by linear **difference** equations with constant coefficients, e.g.,

$$y_k + \alpha y_{k-1} + \beta y_{k-2} = u_k$$

# The Kronecker Delta signal and response

- $\{\delta_k\}_{k\geq 0} = \{1, 0, 0, \ldots\}$ and $\delta_k = 0$ for $k < 0$.
- much easier to understand than the (Dirac) pulse function $\delta(t)$...
- every input signal $\{u_k\}$ to an LTIS can be written as

$$u_k = \sum_{n=-\infty}^{\infty} \delta_n u_{k-n} \text{ for every } k$$

- by the superposition principle (linearity and time invariance), the output signal $\{y_k\}$ satifies

$$y_k = \sum_{n=-\infty}^{\infty} g_n u_{k-n} \text{ for every } k$$

  where $\{g_k\}$ is the response of the LTIS to a delta input
- $\{g_k\}$ is called the delta response of the LTIS
- $\{g_k\}$ is equivalent to the difference equation in defining/decribing the LTIS



# Discrete Convolution

$$y_k = \sum_{n=-\infty}^{\infty} g_n u_{k-n} \text{ for all } k \tag{1}$$

is a **discrete convolution**, denoted $y = g \star u$

- Note the analogy with the continuous convolution $y(t) = \int_{-\infty}^{\infty} g(\tau)u(t-\tau)d\tau$
- Convolution is commutative: $\sum_{n=-\infty}^{\infty} g_n u_{k-n} = \sum_{n=-\infty}^{\infty} g_{k-n} u_n$

  To prove, easiest to go back to the previous slide and note that $u_k = \sum_{n=-\infty}^{\infty} \delta_n u_{k-n} = \sum_{n=-\infty}^{\infty} \delta_{k-n} u_n$.

## Discrete Convolution: special cases

- For semi-infinite sequences $y = \{g_k\}_{k\geq 0} \star \{u_k\}_{k\geq 0}$ simplifies to

$$y_k = \sum_{n=0}^{k} g_n u_{k-n}$$

because $g_n = 0$ for $n < 0$ and $u_{k-n} = 0$ for $n > k$.

- For a finite sequence $\{g_k\}_{0\leq k\leq \ell}$

$$y_k = \sum_{n=0}^{\min(\ell,k)} g_n u_{k-n}$$

- All these special cases are due to the properties of the sequences involved. The definition with a sum from $-\infty$ to $\infty$ remains valid in all cases.



## Properties of the Delta response

- the delta response is also historically called "pulse response" or "impulse response" even though $\{\delta_k\}$ is not really an "impulse" like its continuous time equivalent $\delta(t)$
- **Finite Impulse Response (FIR):** $\{g_k\}_{k\geq 0} = \{g_0, g_1, \ldots, g_\ell, 0, 0, 0, \ldots\}$
- **Infinite Impulse Response (IIR):** $\{g_k\}_{k\geq 0} = \{g_0, g_1, g_2, g_3, \ldots\}$ doesn't terminate
- If $\{g_k\}_{k\geq 0}$ is a semi-infinite sequence, i.e., $g_k = 0$ for $k < 0$, then

$$y_k = \sum_{n=0}^{k} g_n u_{k-n},$$

the **current output only depends on current and past inputs.**

The LTIS is **causal**

# Causality in practice

- Systems that operate as a function of time and in real time are always causal.
- It is wrong to say that "real world systems are always causal".
- For example,
  - in **image processing** the sequence index $k$ often relates to pixel location rather than time and such systems can be non-causal;
  - for signal processing system that operate on **stored or buffered data**, it is also no problem to work with non-causal systems and sometimes offers significant advantages.
  - signal storage and buffering is one of the reasons digital signal processing is so much more powerful than continuous signal processing. Storing or buffering a continuous time signal is not possible in practice, wheras it is commonplace for discrete-time signals.
- **control engineers** focus on systems that must operate in real time and hence for them causality is an essential condition.



# Next lectures

- introduce the $z$ transform. This is the discrete-time equivalent to the Laplace transform
- re-visit the Discrete-Time Fourier Transform (DTFT). This is the discrete-time equivalent of the Fourier transform
- Give us the tools to tackle discrete-time signals and systems as we have learned to tackle continuous-time signals and systems